ANALYSIS OF INTERNAL SUPPORT MOMENTS OF CONTINUOUS BEAMS OF EQUAL SPANS USING SIMPLIFIED MATHEMATICAL MODEL APPROACH

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ABSTRACT
This paper developed the mathematical Model for the evaluation of the internal support moments of a uniformly loaded continuous beam of equal span and the number of spans, taking the uniformly distributed load on the beam to be equal for all spans. The internal support moments of a continuous beam with unit uniformly distributed load (UDL) and unit span length were evaluated at different number of spans (n). The result showed that the values of the internal support moments oscillate (alternately increase and decrease) with rapidly diminishing magnitude at every unit increase in the number of spans (n) of the continuous beam. This showed that it is possible to express the values of the internal support moments with a model. The Euler method for linear interpolation was applied repeatedly between subsequent values of the moments at each internal support for different values of number of spans (n) and each internal moment expressed as a series comprising the sum of their initial values and the subsequent increases and decreases to their values as a result of an increase in the number of spans of the continuous beam. The results obtained showed perfect and close comparison with that obtained using Clapeyron's Method as depicted on the graphs shown in the appendix.
Keywords: Continuous beams, equal Spans, Internal Support moment, Mathematical Model.

INTRODUCTION
Continuous beams can be defined as horizontal structural members which are supported on more than two supports (Anand, 1990). When they are simply supported at their ends, the moments at the two external supports are zero.

In Structural Engineering, most especially under reinforced concrete design, the engineer will often engage in the analyses of continuous beams. This is done as a simplification of the analyses as often times, reinforced concrete structures are framed structures (Hibbeller, 2006). Their design as continuous beams reduces their degrees of statical indeterminacy and facilitates easier analyses (BS 8110, 1997; Mosley and Bungey, 1999). This simplification though, a considerate compromise, seemed inadequate as most continuous beams are indeterminate and their degrees of indeterminacy increases by two units for every additional support introduced. Their analyses involve the generation and solution of many simultaneous equations which are tedious by hand calculation and prone to errors. In an apparent bid to lend a helping hand, table 3.5 of BS 8110 Part 1, 1997 further provided approximate values of support and span moments to be used in designing continuous beams of approximately equal spans. These efforts to simplify the process of analyzing continuous beams do so at the expense of the accuracy of the results, hence the need to build a mathematical model for analyzing continuous beams that is easy to evaluate and with precise results.

Formulation of the Mathematical Model

Continuous beams can be analyzed by different methods; the two most common methods are moment distribution method and Clapeyron's three moment equation method. The moment distribution method is an iterative approach; its accuracy depends on the number of cycles of iteration carried out. This method is easier by hand calculation but the results are less precise. The three moment method though more tedious than the moment distribution method (because it involves the solution of simultaneous equations), gives precise solutions. In this research the Clapeyron's three moment equation method was adopted in developing the mathematical model because it gives precise values. According to Righiniotis (2007), the Clapeyron's theorem for the first three supports of a homogenous beam can be stated as

$$M_1\frac{L_1}{I_1}+2M_2\left(\frac{L_1}{I_1}+\frac{L_2}{I_2}\right)+M_3\frac{L_2}{I_2}=6\left(\frac{A_1x_1}{L_1I_1}+\frac{A_2x_2}{L_2I_2}\right) \quad \text{...} (1)$$

Where, M_1, M_2 and M_3 are the numerical values of the hogging moments at the first three supports.

A_1 is the area of the free bending moment diagram between the first and second support

A_2 the area of the free bending moment diagram between the second and third support.

x_1 and x_2 are the distances from the centroid of the free bending moment diagrams, A_1 and A_2 from the support.

I_1 and I_2 are the second moment of area of the members' cross section.

When the entire continuous beam is of the same cross section $I_1=I_2=I$ and so I cancels out from the equation.

For a case of uniformly distributed load w on a span

$$6\frac{A_1x}{L_1}=\frac{wl^3}{4} \qquad \text{(Oyenuga, 2001)}$$

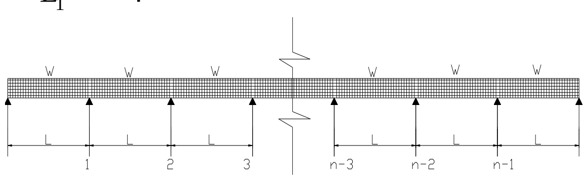

Figure 1: Uniformly loaded Continuous beam of n spans

In the analysis of the beam in *figure 1*, the resulting simultaneous equations expressed in matrix form is $KM = L$ which is

$$\begin{bmatrix} 4L & L & 0 & 0 & \cdots & 0 & 0 & 0 \\ L & 4L & L & 0 & \cdots & 0 & 0 & 0 \\ 0 & L & 4L & L & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 4L & L & 0 \\ 0 & 0 & 0 & 0 & \cdots & L & 4L & L \\ 0 & 0 & 0 & 0 & \cdots & 0 & L & 4L \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{wl^3}{2} \\ \frac{wl^3}{2} \\ \frac{wl^3}{2} \\ \vdots \\ \frac{wl^3}{2} \\ \frac{wl^3}{2} \\ \frac{wl^3}{2} \end{bmatrix} \qquad \text{.. (2)}$$

$$M = K^{-1}L \quad \text{.. (3)}$$

$$M^T = \begin{bmatrix} M_1 & M_2 & M_3 & \cdots & M_{n-3} & M_{n-2} & M_{n-1} \end{bmatrix} \text{.. (4)}$$

K is a square matrix and has all its elements equal to zero except of a band centred on the main diagonal, hence it is a banded matrix. (Chapra et al, 2007)

Equation (3) was evaluated for $w = 1kN/m^2$, $L = 1m$ and number of spans $n = 2$ *to* 15 and their values printed in table 1 in Appendix A. The Euler method was then applied repeatedly between the moment values for successive number of spans to produce the required series.

New value = old value + slope x *step* ..Euler method

$$f(x+1) = f(x) + \frac{f(x+1) - f(x)}{n_1 - n_0} \quad \text{...... (5)}$$

Likewise

$$f(x+2) = f(x+1) + \frac{f(x+2) - f(x+1)}{n_2 - n_1} \quad \text{......... (6)}$$

Substituting equation (5) into (6) will give

$$f(x+2) = f(x) + \frac{f(x+1) - f(x)}{n_1 - n_0} + \frac{f(x+2) - f(x+1)}{n_2 - n_1} \text{....................................... .... (7)}$$

Continuing further we will arrive at a general formula

$$f(x+a) = f(x) + \frac{f(x+1) - f(x)}{n_1 - n_0} + \frac{f(x+2) - f(x+1)}{n_2 - n_1} + \frac{f(x+3) - f(x+2)}{n_3 - n_2} + \cdots + \frac{f(x+a) - f(x+a-1)}{n_a - n_{a-1}} \quad \text{.. (8)}$$

If n = number of spans, $n_1 = 1, n_2 = 2, n_3 = 3 \ldots n_n = n$

i = internal support moment number, $i = 1, 2, 3 \ldots n-1$.

$M_{i,n}$ = Moment at the i^{th} internal support for n-spanned continuous beam

$$M_{i,n} = M_{i,1} + \frac{M_{i,2} - M_{i,1}}{n_2 - n_1} + \frac{M_{i,3} - M_{i,2}}{n_3 - n_2} + \frac{M_{i,4} - M_{i,3}}{n_4 - n_3} + \cdots + \frac{M_{i,n} - M_{i,n-1}}{n_n - n_{n-1}} \quad \text{.. (9)}$$

Where $M_{i,n}$ is the i[th] internal support moment of spanned continuous beam.

It is observed that $M_{i,i} = 0$ and can be dropped because a simply supported beam has no internal support. Likewise a two spanned beam has no second interior support and so on.

Each term in equation (9) was evaluated for each internal support moment M_i, i = 1, 2, 3… 10 using the values in table 1.
Later the variation of the values of each term (in the series) with an increase or decrease in the uniformly distributed load w and length l of span were investigated by comparing values and this enabled the determination of the exponents of the uniformly distributed load(w) and the length of span(l) in each term of the series.

Results and Discussion

From the tables, it can be seen that each internal support moment can be expressed in the form of equation (8). Since each term is introduced because of an additional span added to the beam, not all the terms in the series need always to be evaluated to determine the internal support moment M_i. For an n-spanned continuous beam

$M_i = \sum_{i}^{n-1} terms$ (Excluding $M_{i,i}$ that is equal to zero)

Where n = number of spans
M_i = the i[th] internal support moment
To eliminate the mistake of evaluating more terms than required the function {X} was introduced.
{X}= 0 when $X \leq 0$
{X}= 1 when $X > 0$

The series for evaluating the internal support moment of continuous beams of equal spans and uniformly distributed load are summarized below.

$$M_1 = wl^2\left[\begin{array}{l}\frac{\langle n-1\rangle}{8}-\frac{\langle n-2\rangle}{40}+\frac{\langle n-3\rangle}{140}-\frac{\langle n-4\rangle}{532}+\frac{\langle n-5\rangle}{1976}-\frac{\langle n-6\rangle}{7384}\\+\frac{\langle n-7\rangle}{27548}-\frac{\langle n-8\rangle}{102820}+\frac{\langle n-9\rangle}{383720}-*\end{array}\right] \quad \text{................ (10.1)}$$

$$M_2 = wl^2\left[\begin{array}{l}\frac{\langle n-2\rangle}{10}-\frac{\langle n-3\rangle}{35}+\frac{\langle n-4\rangle}{133}-\frac{\langle n-5\rangle}{494}+\frac{\langle n-6\rangle}{1846}-\frac{\langle n-7\rangle}{6887}\\+\frac{\langle n-8\rangle}{25705}-\frac{\langle n-9\rangle}{95930}+\frac{\langle n-10\rangle}{358018}-*\end{array}\right] \quad \text{................................ (10.2)}$$

$$M_3 = wl^2\left[\begin{array}{l}\frac{3\langle n-3\rangle}{28}-\frac{15\langle n-4\rangle}{532}+\frac{15\langle n-5\rangle}{1976}-\frac{15\langle n-6\rangle}{7384}+\frac{15\langle n-7\rangle}{27548}-\frac{3\langle n-8\rangle}{20564}\\+\frac{3\langle n-9\rangle}{76744}-\frac{2\langle n-10\rangle}{190943}+\frac{\langle n-11\rangle}{356304}+*\end{array}\right] \quad \text{................................ (10.3)}$$

$$M_4 = wl^2\left[\begin{array}{l}\frac{2\langle n-4\rangle}{19}-\frac{7\langle n-5\rangle}{247}+\frac{7\langle n-6\rangle}{923}-\frac{14\langle n-7\rangle}{6887}+\frac{14\langle n-8\rangle}{25705}-\frac{7\langle n-9\rangle}{47965}\\+\frac{7\langle n-10\rangle}{179009}-\frac{2\langle n-11\rangle}{190877}+\frac{2\langle n-12\rangle}{712363}-*\end{array}\right] \quad \text{......... (10.4)}$$

$$M_5 = wl^2\left[\begin{array}{l}\frac{11\langle n-5\rangle}{104}-\frac{209\langle n-6\rangle}{7384}+\frac{47\langle n-7\rangle}{6195}-\frac{26\langle n-8\rangle}{12791}+\frac{52\langle n-9\rangle}{95471}-\frac{52\langle n-10\rangle}{356305}\\+\frac{26\langle n-11\rangle}{664873}-\frac{5\langle n-12\rangle}{477181}-\frac{\langle n-13\rangle}{356173}+*\end{array}\right] \quad \ldots\ldots\ldots\ldots (10.5)$$

$$M_6 = wl^2\left[\begin{array}{l}\frac{15\langle n-6\rangle}{142}-\frac{195\langle n-7\rangle}{6887}+\frac{39\langle n-8\rangle}{5141}-\frac{39\langle n-9\rangle}{19186}+\frac{97\langle n-10\rangle}{178091}-\frac{\langle n-11\rangle}{6852}+\\\frac{\langle n-12\rangle}{25572}-\frac{\langle n-13\rangle}{95436}+\frac{\langle n-14\rangle}{356172}-*\end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.6)$$

$$M_7 = wl^2\left[\begin{array}{l}\frac{41\langle n-7\rangle}{388}-\frac{165\langle n-8\rangle}{5828}+\frac{93\langle n-9\rangle}{12259}-\frac{21\langle n-10\rangle}{10331}+\frac{73\langle n-11\rangle}{134027}-\frac{\langle n-12\rangle}{6852}\\+\frac{\langle n-13\rangle}{25572}-\frac{\langle n-14\rangle}{95436}+\frac{\langle n-15\rangle}{356172}-*\end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots (10.7)$$

$$M_8 = wl^2\left[\begin{array}{l}\frac{28\langle n-8\rangle}{265}-\frac{128\langle n-9\rangle}{4521}+\frac{11\langle n-10\rangle}{1450}-\frac{41\langle n-11\rangle}{20170}+\frac{78\langle n-12\rangle}{143207}-\frac{\langle n-13\rangle}{6852}\\+\frac{\langle n-14\rangle}{25572}-\frac{\langle n-15\rangle}{95436}+\frac{\langle n-16\rangle}{356172}-*\end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.8)$$

$$M_9 = wl^2\left[\begin{array}{l}\frac{153\langle n-9\rangle}{1448}-\frac{287\langle n-10\rangle}{10137}+\frac{302\langle n-11\rangle}{39809}-\frac{41\langle n-12\rangle}{20170}+\frac{76\langle n-13\rangle}{139535}-\frac{\langle n-14\rangle}{6852}\\+\frac{\langle n-15\rangle}{25572}-\frac{\langle n-16\rangle}{95436}+\frac{\langle n-17\rangle}{356172}-*\end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.9)$$

$$M_{10} = wl^2\left[\begin{array}{l}\frac{209\langle n-10\rangle}{1978}-\frac{78\langle n-11\rangle}{2755}+\frac{379\langle n-12\rangle}{49950}-\frac{41\langle n-13\rangle}{20170}+\frac{77\langle n-14\rangle}{141371}-\frac{\langle n-15\rangle}{6852}\\+\frac{\langle n-16\rangle}{25572}-\frac{\langle n-17\rangle}{95436}+\frac{\langle n-18\rangle}{356172}-*\end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (10.10)$$

* value is less than 1 x 10^{-6} and can be ignored

Graphs of the values of equations (10.1) to (10.10) above (connected with continuous line) and the values of the internal support moments (By the Clapeyron's three moment equation) represented with 'o' plotted against number of spans (n) are shown in *figure 2* to 10 in Appendix B. It can be seen that equations (10.1) to 10.10) are good representation of the internal support moments. The errors though not visible in the graphical representation of figure 2 are due to approximation and the truncated terms depicted with '*'in the equations. The error is thus less than or equal to 1 x $10^{-6}wl^2$ as this was the criterion used in the truncation. As *n* => 24, the internal moments, starting with the most interior supports, $M_{i = n/2 \text{ or } (n-1)/2} \geq wl^2/12$

CONCLUSION

The analysis of continuous beams is a lot easier when evaluated with the mathematical model formulated with finite series. This will help reduce the time spent on the structural analyses of long continuous beams. It is however important to note that the above series were developed for continuous beams with constant uniformly distributed load and equal span. If the span or uniformly distributed loads of a continuous beam vary significantly, the series may not evaluate correctly. A computer program for the calculation of these internal support moments were developed using equations (10.1) – (10.10). The results obtained were observed to exhibit perfect comparison with that of clapeyron's method.

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APPENDIX A

Table 1

Internal Supports Moments for an n-spanned continuous beam

Max No of Spans = 15 w = 1kN/m l = 1m

n	2	3	4	5	6	7	8	9	10
M1	1/8	1/10	3/28	2/19	11/104	15/142	41/388	28/265	153/1448
M2	-	1/10	1/14	3/38	1/13	11/142	15/194	41/530	14/181
M3	-	-	3/28	3/38	9/104	6/71	33/388	9/108	123/1443
M4	-	-	-	2/19	1/13	6/71	8/97	22/265	15/181
M5	-	-	-	-	11/104	11/142	33/388	22/265	121/1448
M6	-	-	-	-	-	15/142	15/194	9/106	15/181
M7	-	-	-	-	-	-	41/388	41/530	123/1448
M8	-	-	-	-	-	-	-	28/265	14/181
M9	-	-	-	-	-	-	-	-	153/1448

n	11	12	13	14	15
M1	209/1978	571/5404	390/3691	390/3691	390/3691
M2	153/1978	209/2702	195/2521	195/2521	195/2521
M3	84/989	459/5404	340/4003	234/2755	234/2755
M4	82/989	16/193	306/3691	209/2521	193/2328
M5	165/1978	451/5404	308/3691	541/6483	121/1450
M6	165/1978	225/2702	307/3685	210/2521	229/2749
M7	82/989	451/5404	307/3685	420/5039	1148/13775
M8	84/989	16/193	308/3691	210/2521	1148/13775
M9	153/1978	459/5404	306/3691	541/6483	230/2761
M10	209/1978	209/2702	340/4003	209/2521	121/1450
M11	-	571/5404	195/2421	234/2755	193/2328
M12	-	-	390/3691	195/2521	234/2755
M13	-	-	-	390/3691	195/2521
M14	-	-	-	-	390/3691

Mi = Moment at the i^{th} internal support
n = number of spans.
l = length of span

APPENDIX B

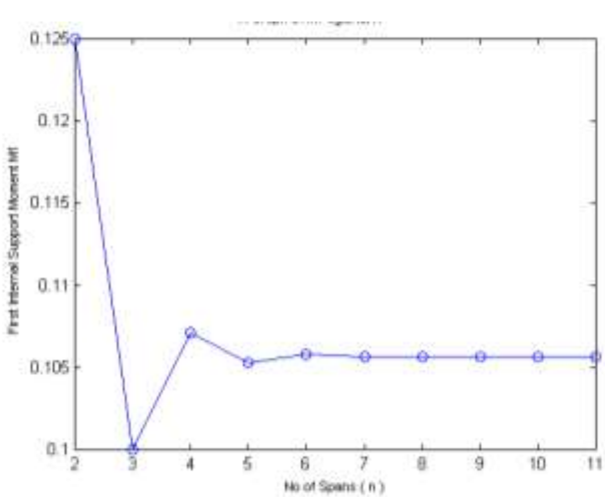


Figure 2. – A graph of internal support moment M_1 against number of spans (n)

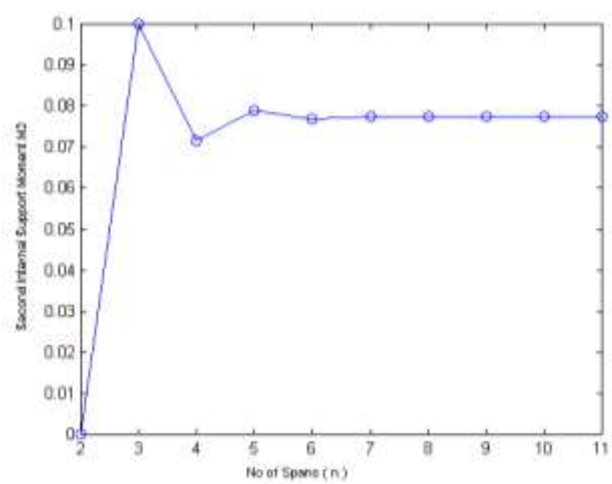


Figure 3 - A graph of internal support moment M_2 against number of spans (n)

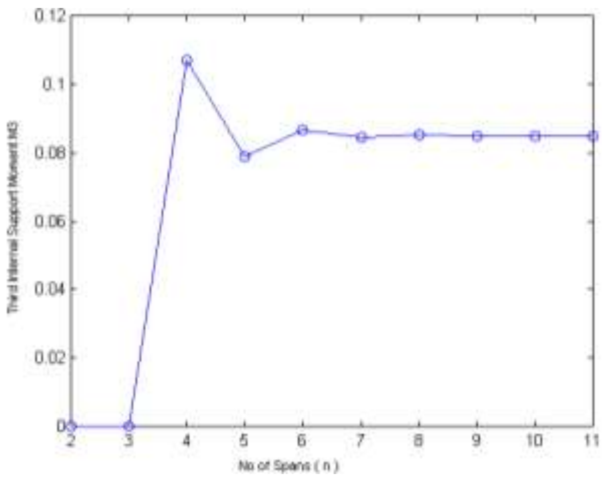


Figure 4 - A graph of internal support moment M_3 against number of spans (n)

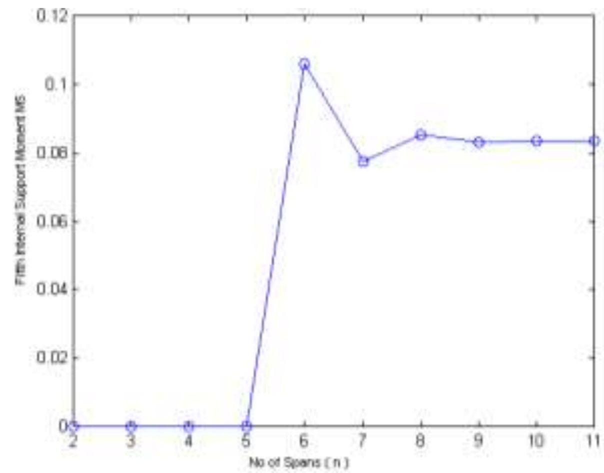
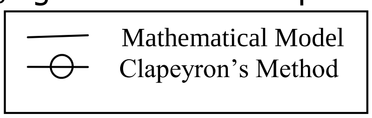


Figure 6 - A graph of internal support moment M_5 against number of spans (n)



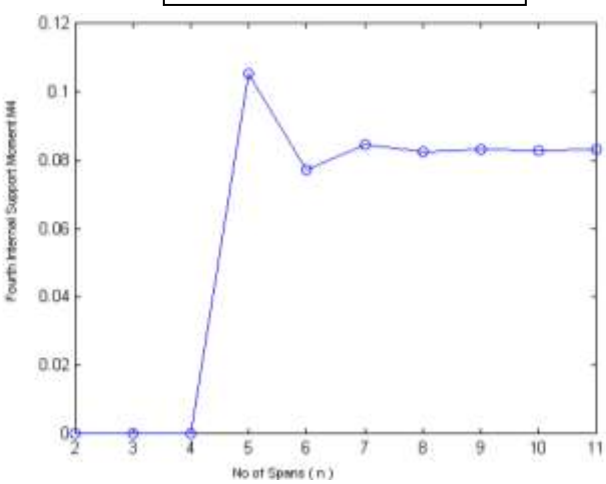


Figure 5 - A graph of internal support moment M_4 against number of spans (n)

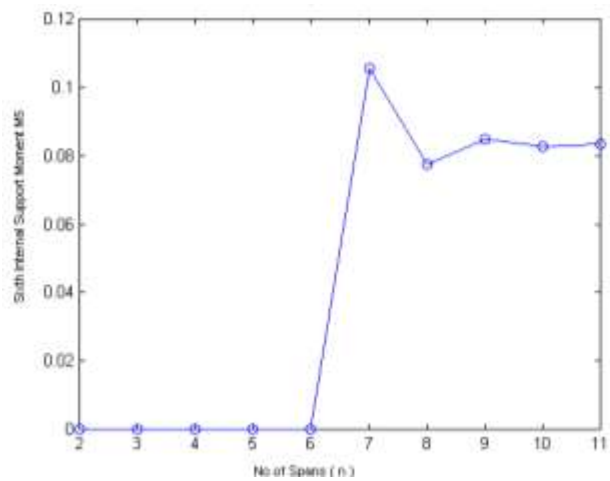


Figure 7 - A graph of internal support moment M_6 against number of spans (n)

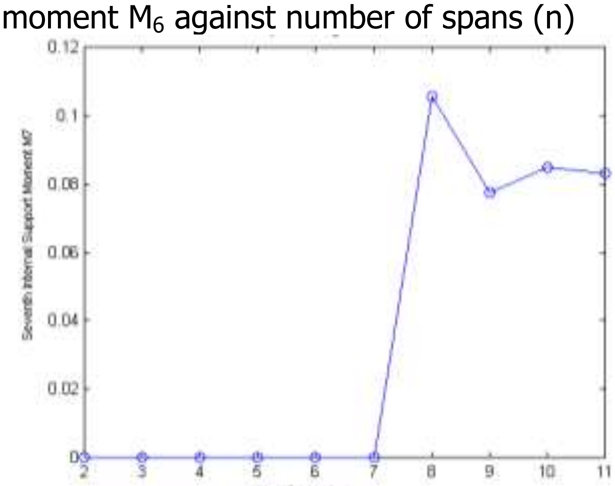


Figure 8 - A graph of internal support moment M_7 against number of spans (n)

Mathematical Model
Clapeyron's Method

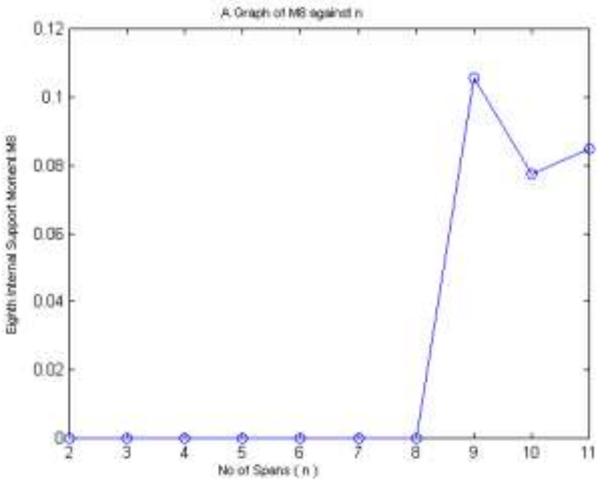


Figure 9- A graph of internal support moment M_8 against number of spans (n)

Mathematical Model
Clapeyron's Method

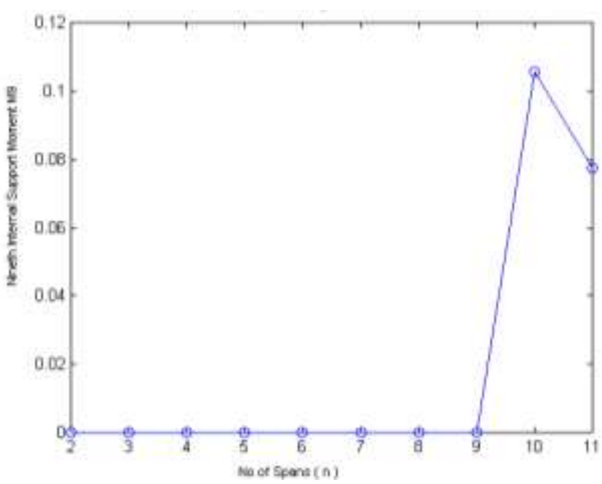


Figure 10 - A graph of internal support moment M_9 against number of spans (n)

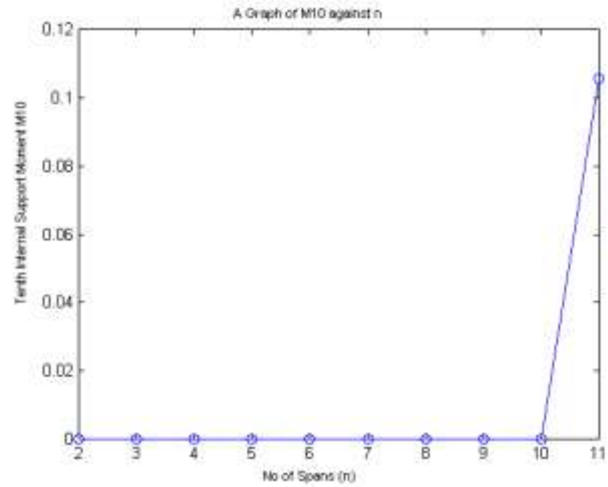


Figure 11- A graph of internal support moment M_{10} against number of spans (n)

Mathematical Model
Clapeyron's Method

LEGEND