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SOME EXACT SOLUTIONS OF BOUNDARY LAYER FLOW IN POROUS MEDIA-II

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ABSTRACT

Some exact solutions are presented for the unsteady boundary layer flows of a homogenous, viscous, incompressible fluid bounded by (i) an infinite rigid oscillating flat plate or (ii) two parallel rigid oscillating flat plates as presented in [7] was extended to porous media. An explicit representation of the velocity fields for both the configurations has been given. The structures of the associated periodic boundary layers are determined with physical interpretations as in [4]. Several results of interest have been recovered as special cases of this general theory. The Heaviside operational calculus along with the theory of residues of analytic functions is adopted in finding the solutions.

INTRODUCTION

In the earlier works [1], unsteady boundary layer flows generated in a homogenous, incompressible viscous fluid by moving the boundary of the fluid impulsively with a prescribed velocity. The Laplace transform method [2] has been used to obtain exact solutions of the unsteady boundary layer equations in some general configurations. The structures of the unsteady velocity field and the associated boundary layers have been determined with physical interpretations. In addition to the generalizations of the earlier results, some new results of interest have been found. Basant and Kaurangini [8] studied unsteady boundary layer flows in porous media generated in a homogenous, non-rotating viscous fluid. The method of Laplace transform is used to obtain exact solutions of the unsteady boundary layer flow in a porous medium in a more general situation. The structures of the unsteady velocity field and associated boundary layers are determined. Several particular solutions are presented as special cases of the present general solutions. The physical implications of the mathematical results are investigated.

In this paper, we are primarily interested in extending the work of Michael and Lokenath [7] to the porous medium. This is the study of the unsteady periodic boundary layer phenomena in a homogeneous incompressible viscous fluid in porous media. Some general and exact solutions of the unsteady boundary layer equation are presented for two geometric configurations as in [1] and [7]. It is shown that the ultimate steady solution consists of double stokes layers of thicknesses of the orders $\sqrt{\nu/\omega_r}$ $(\mathrm{r}=1,2)$ where ν is the kinematics viscosity of the fluid and ω_r represents the forcing frequency of the boundary of the fluid or the fluid in the porous media. The results of the earlier investigators are recovered as special cases of the present general result. In addition to the extensions of the earlier results, some new results of interest are obtained with their physical significance explored. The Heaviside operational calculus along with the theory of residues of analytic functions is employed for the investigation of the problems.

EQUATION AND GENERAL SOLUTION

We consider the periodic unsteady boundary layer flows engendered in porous media are considered (i) in a semi-infinite expanse of a homogeneous non-rotating viscous fluid bounded by an infinite horizontal plate and (ii) in a homogeneous viscous fluid between two infinite parallel rigid flat plates. The flow is generated in configuration (i) by the oscillations of the plate and the oscillations of the main flow outside the boundary layer. In configuration (ii), both the boundary plates perform elliptic harmonic oscillations in their own planes so that an unsteady flow is set up in the fluid. In the two-dimensional, Cartesian coordinates is employed with the origin, x-axis and y-axis in the plate at z=0 and the z-axis vertically normal to the plate. With the coordinate system, the unsteady flow is governed by the Navier-Stokes equation and the continuity equation in the forms

$$\frac{\partial u}{\partial t}+(u.\nabla)u=-\frac{1}{\rho}\nabla P+\nu_{eff}\nabla^2 u-\frac{\upsilon u}{k}, \qquad (2)$$

$$div \;\; \mathrm{u}=0,$$

(3)

Where $u=(u,v,w)$ is the velocity field, ρ is the density, and P is the pressure.

In view of the symmetry of the configurations, all the quantities depend only on z and time t based on the usual assumptions of the unsteady two-dimensional flow, it can readily be seen that the non-linear term $(u.\nabla)u$ disappears exactly from (2). Consequently, it follows from (2) and (3) that the equations of the boundary layer flow in porous media take the form

$$-P=\frac{\partial u}{\partial t}+\frac{\nu u}{k}, \qquad (4)$$

Where $P\equiv(1/\rho)(\partial P/\partial x)$ is the pressure gradient.

The boundary conditions for configuration (i) are

$$u(z,t)=f(t) \quad \text{on } z=0 \quad t>0, \qquad (5)$$

and $u(z,t)\to g(t)$ as $z\to\infty$, $t>0$, (6)

where $f(t)$ and $g(t)$ arbitrary functions of t and their particular forms are specified later.

For the flow in the inviscid region to be consistent with the basic equations of motion, we require $-\mathrm{P}=\frac{\mathrm{dg(t)}}{\mathrm{dt}}+\frac{\upsilon\mathrm{g(t)}}{\mathrm{k}}$ (7)

Where u is given by (6)

The initial condition is

$$u(z,t)=0 \quad \text{at} \quad t<0 \quad \text{for all } z \qquad (8)$$

This initial value problem can be readily solved by introducing the Laplace transform defined by the integral

$$\bar{u}(z,s)=\int_0^\infty e^{-st}u(z,t)dt \qquad (9)$$

The inverse Laplace transformation is given by

$c > 0$

Using the Laplace transform method and initial condition (8), the solution for $u(z,t),$ subject to the required boundary conditions, can be

$$u(z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\bar{f}(s)\exp(-\lambda z) + \bar{g}(s)(1-\exp(-\lambda z))\right]\exp(st)\,ds \qquad (11)$$

Where $\lambda = \sqrt{\left(\frac{\upsilon}{k} + s\right)\Big/\upsilon_{eff}}$ (12)

By virtue of the convolution theorem for the Laplace transform, solution (11) reduces to

$$u(z,t) = g(t) + \frac{z\exp(-\frac{\upsilon t}{k})}{2\sqrt{\pi\upsilon_{eff}}}\int_0^t\left[f(t-\tau) - g(t-\tau)\right]\tau^{-3/2}\exp\left(-\frac{z^2}{4\upsilon_{eff}\,\tau}\right)d\tau \qquad (13)$$

This is the most general as well as exact solution of the velocity distribution. To investigate the flow features and the structures of the associated boundary layers, it is of interest to consider some special cases. We consider the following particular

We consider the following particular case

$f(t) = ae^{i\omega_1 t} + be^{-i\omega_1 t},$

$f(t) = a\exp(i\omega_1 t) + b\exp(-i\omega_1 t)$ and (14a, b)

$g(t) = ce^{i\omega_2 t} + de^{-i\omega_2 t}$

Where ω_1, ω_2 .are the given frequency of oscillations, a, b, c, and d are complex constants of order one. This case includes several other special cases of interest which arise when

1. $b = 0$ and $c = d = 0$
2. $c = d = 0$
3. $f(t) = a + be^{i\omega_1 t}$ *and* $c = d = 0$
4. $a = b = 0$ and $g(t) = c + d\exp(i\omega_2 t)$
5. $a = b = 0$ and $g(t) = c\exp(i\omega_2 t) + d\exp(-i\omega_2 t)$
6. $f(t) = ae^{i\omega_1 t}$ *and* $g(t) = c$
7. $\omega_1 = \omega_2$

DISCUSSION AND CONCLUSION

$$u(z,t)=\frac{a\sinh\left((D-z)\sqrt{\left(\frac{i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}{\sinh\left(D\sqrt{\left(\frac{i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}e^{i\omega_1 t}+\frac{b\sinh\left((D-z)\sqrt{\left(\frac{-i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}{\sinh\left(D\sqrt{\left(\frac{-i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}e^{-i\omega_1 t}$$

$$+\frac{cSinh\left(z\sqrt{\frac{\left(\frac{i\omega_2}{\nu}+\frac{1}{k}\right)}{\gamma}}\right)}{\sinh\left(D\sqrt{\left(\frac{i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}e^{i\omega_2 t}+\frac{d\sinh\left((z)\sqrt{\left(\frac{-i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}{\sinh\left(D\sqrt{\left(\frac{-i\omega_1}{\nu}+\frac{1}{k}\right)\Big/\gamma}\right)}e^{-i\omega_2 t}$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{2\pi n\nu a\gamma\sin\left(\left(n\pi\left(1-\frac{z}{D}\right)\right)\right)}{\left[n^2\pi^2+\frac{D^2}{k\gamma}+i\omega_1 D^2\right]}e^{-\delta t}+\sum_{n=1}^{\infty}(-1)^n\frac{2\pi n\nu b\gamma\sin\left(\left(n\pi\left(1-\frac{z}{D}\right)\right)\right)}{\left[n^2\pi^2+\frac{D^2}{k\gamma}-i\omega_1 D^2\right]}e^{-\delta t}$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{2\pi n\nu c\gamma\sin\left(\left(n\pi\frac{z}{D}\right)\right)}{\left[n^2\pi^2+\frac{D^2}{k\gamma}+i\omega_2 D^2\right]}e^{-\delta t}+\sum_{n=1}^{\infty}(-1)^n\frac{2\pi n\nu d\gamma\sin\left(\left(n\pi\frac{z}{D}\right)\right)}{\left[n^2\pi^2+\frac{D^2}{k\gamma}-i\omega_2 D^2\right]}e^{-\delta t}$$

(15)

Where $\delta=\left(\frac{n^2\pi^2}{D^2}+\frac{1}{k\gamma}\right)\nu\gamma$

(16)

It is evident from (15) that the velocity field consists of both steady state and transient components. In the limit $t\to\infty$, the last four infinite series representing the transient solution decay exponentially to zero. Consequently, the ultimate steady state is attained in the limit.

e steady-state solution
boundary layers on both the plates. These layers have depths of penetration of oscillations of the order $\left(\nu/\omega_1\right)^{1/2}$ on z=0 and $\left(\nu/\omega_2\right)^{1/2}$ on z=D. solution (15) includes several special cases of interest.

In particular, when $f(t) \equiv a + be^{i\omega_1 t}$ and $g(t) \equiv c + de^{i\omega_2 t}$, the velocity field $u(z,t)$ can be found in the form. In the limit $t \to \infty$, the transient terms die out and the solution approaches the ultimate steady state.

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