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APPLICATION OF FINITE SERIES IN THE ANALYSIS OF INTERNAL SUPPORT MOMENTS OF CONTINUOUS UNIFORMLY LOADED BEAMS OF EQUAL SPANS

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ABSTRACT

This paper developed a mathematical Model for evaluating the interior support moments of continuous beams. The interior support moment is a function of the load w; span length l; number of spans n and flexural rigidity EI. $M = f(n, w, l, EI)$. By keeping the variables w, l, and EI constant (i.e equal to 1), the mathematical relationship between the internal support moment M and number of spans n was established. The internal support moments were expressed as a finite series with terms dependent on n. Next the exponent of w and l in each term of the series was determined to be one and two respectively; this was done by varying their values alternately and checking the contribution of each variation on each term in the series. Finally the series was developed for a unit value of w on one span and zero value on other spans and repeated for all others spans; this leads to interesting results and by superposition principle, each internal support moment was expressed as the sum of the corresponding series for each case. The results were found to compare favourably with that obtained from Hardy Cross method of moment distribution and the Clapeyron's three moments method.
Keywords: Finite series, internal support, Moments, span, uniformly distributed load.

INTRODUCTION

Analysis of continuous beams is a common part of the design process. In reinforced concrete structures, the load bearing skeleton of the building is very often a frame, but to simplify the process of analysis by hand, BS 8110 allows the frame to be split and analysed as continuous beams. In steel design the roof rafters support a uniformly spaced point loads from the purlins and can be analysed as a continuous beam. BS 8110 allows one way spanning slabs to be analysed as a continuous slab, combined footings, floor beams, lintels etc all present avenues for analyses as continuous members.

The most general method for the analyses of beams is the use of the Clapeyron's theorem of three moments which gives the internal support moments of continuous beams and once these are known the shear forces, maximum span moment, point of contraflexure etc can be well delineated (Mosley and Bungey, 1999). These bending moments and shear forces govern the sizes, proportioning and reinforcement of load bearing elements in a structure (BS 5950, 1985; BS 8110, 1997; Oyenuga, 2001, MacGinley et al, 1997).

Model

The internal support moment of uniformly loaded continuous beams of constant flexural rigidity depends on the number of spans n, uniformly distributed loads w_1, w_2, w_3… w_n and span l. Expressed mathematically, it is

$M = f(n, w_1, w_2, w_3, \cdots, w_n, l)$ .... (1)
Where M is an internal support moment, n is the number of spans, w_1 is the uniformly distributed load on the first span, w_2 is the uniformly distributed load on the second span, w_n is the uniformly distributed load on the nth span. l is the span.
To reduce the number of variables to a manageable size all the uniformly distributed loads (UDLs) were made equal i.e. $w_1 = w_2 = w_3 = \ldots = w_n = w$ so that equation (1) can be rewritten as
$M = f(n, w, l)$ .... (2)
Where w is the span UDL.

Relationship between Moment (M) and number of span (n); load (w) and length (l) constant.
In other to express M in terms of n the values of w and l are taken to be unity: w = 1, l = 1 and the following two steps taken.
Step 1
The internal support moments, M_i were calculated for the number of spans n = 2, 3, 4… 15. M_i is the i^{th} internal support.
Example for n = 2 M_1 was calculated
For n = 3 M_1 and M_2 were calculated
For n = 4 M_1, M_2 and M_3 were calculated and so on.
The values of the internal support moments M_1, M_2, M_3… M_{n-1} for n = 2, 3, 4… 15 are printed in table 1. A plot of M_1 against n is shown in fig 1, which is a zigzag line and shows that M_1 can be expressed as a series. Other interior support moments exhibited the same behaviour.

Step 2
According to the Euler's method (Chapra et al, 2007)

$New\ value = old\ value + slope \times step$ .... (3)
For the purpose of this work
$f(x+1) = f(x) + \frac{f(x+1) - f(x)}{1} \cdot 1$ .... (4)
$f(x+2) = f(x+1) + \frac{f(x+2) - f(x+1)}{1} \cdot 1$ .... (5)
The denominator s are 1 because (x + 1) – x = (x + 2) – (x + 1) = 1
Step size = 1

Substituting (4) into (5)

$f(x+1) = f(x) + \frac{f(x+1) - f(x)}{1} + \frac{f(x+2) - f(x+1)}{1}$
.... (6)
By continually repeating the process the equation below is generated.

$f(x+a) = f(x) + \frac{f(x+1) - f(x)}{1} + \frac{f(x+2) - f(x+1)}{1} + \cdots + \frac{f(x+a) - f(x+a-1)}{1}$
.... (7)

In the calculation in step 1, n was increased by 1 after each round hence the step size was one.

Let M_{in} be the moment at the i^{th} internal support for a n-spanned continuous beam. Then by the expression of equation (7), the internal support moments can be written as

$$M_{in} = M_{i1} + \frac{M_{i2} - M_{i1}}{1} + \frac{M_{i3} - M_{i2}}{1} + \cdots + \frac{M_{in} - M_{in-1}}{1}$$
.... (8)

Where M_{in} stands as the i^{th} internal support moment of an n-spanned continuous beam.

By expressing each interior support moment with equation (8) using the values in table 1and putting all M_{ii} = 0, the equations below are generated.

$$M_1 = wl^2\left[\frac{(n-1)}{8} - \frac{(n-2)}{40} + \frac{(n-3)}{140} - \frac{(n-4)}{532} + \frac{(n-5)}{1976} - \frac{(n-6)}{7384} + \frac{(n-7)}{27548} - \frac{(n-8)}{102820} + \frac{(n-9)}{383720} - *\right]$$ .... (9.1)

$$M_2 = wl^2\left[\frac{(n-2)}{10} - \frac{(n-3)}{35} + \frac{(n-4)}{133} - \frac{(n-5)}{494} + \frac{(n-6)}{1846} - \frac{(n-7)}{6887} + \frac{(n-8)}{25705} - \frac{(n-9)}{95930} + \frac{(n-10)}{358018} - *\right]$$ .... (9.2)

$$M_3 = wl^2\left[\frac{3(n-3)}{28} - \frac{15(n-4)}{532} + \frac{15(n-5)}{1976} - \frac{15(n-6)}{7384} + \frac{15(n-7)}{27548} - \frac{3(n-8)}{20564} + \frac{3(n-9)}{76744} - \frac{2(n-10)}{190943} + \frac{(n-11)}{356304} - *\right]$$ .... (9.3)

$$M_4 = wl^2\left[\frac{2(n-4)}{19} - \frac{7(n-5)}{247} + \frac{7(n-6)}{923} - \frac{14(n-7)}{6887} + \frac{14(n-8)}{25705} - \frac{7(n-9)}{47965} + \frac{7(n-10)}{179009} - \frac{2(n-11)}{190877} + \frac{2(n-12)}{712363} - *\right]$$ .... (9.4)

$$M_5 = wl^2\left[\frac{11(n-5)}{104} - \frac{209(n-6)}{7384} + \frac{47(n-7)}{6195} - \frac{26(n-8)}{12791} + \frac{52(n-9)}{95471} - \frac{52(n-10)}{356305} + \frac{26(n-11)}{664873} - \frac{5(n-12)}{477181} + \frac{(n-13)}{356173} - *\right]$$ .... (9.5)

$$M_6 = wl^2\left[\frac{15(n-6)}{142} - \frac{195(n-7)}{6887} + \frac{39(n-8)}{5141} - \frac{39(n-9)}{19186} + \frac{97(n-10)}{178091} - \frac{(n-11)}{6852} + \frac{(n-12)}{25572} - \frac{(n-13)}{95436} + \frac{(n-14)}{356172} - *\right]$$ .... (9.6)

$$M_7 = wl^2\left[\frac{41(n-7)}{388} - \frac{165(n-8)}{5828} + \frac{93(n-9)}{12259} - \frac{21(n-10)}{10331} + \frac{73(n-11)}{134027} - \frac{(n-12)}{6852} + \frac{(n-13)}{25572} - \frac{(n-14)}{95436} + \frac{(n-15)}{356172} - *\right]$$ .... (9.7)

$$M_8 = wl^2\left[\frac{28(n-8)}{265} - \frac{128(n-9)}{4521} + \frac{11(n-10)}{1450} - \frac{41(n-11)}{20170} + \frac{78(n-12)}{143207} - \frac{(n-13)}{6852} + \frac{(n-14)}{25572} - \frac{(n-15)}{95436} + \frac{(n-16)}{356172} - *\right]$$ .... (9.8)

$$M_9 = wl^2\left[\frac{153\,(n-9)}{1448} - \frac{287\,(n-10)}{10137} + \frac{302\,(n-11)}{39809} - \frac{41\,(n-12)}{20170} + \frac{76\,(n-13)}{139535} - \frac{(n-14)}{6852} + \frac{(n-15)}{25572} - \frac{(n-16)}{95436} + \frac{(n-17)}{356172} - *\right]$$
……………………………. (9.9)

$$M_{10} = wl^2\left[\frac{209\,(n-10)}{1978} - \frac{78\,(n-11)}{2755} + \frac{379\,(n-12)}{49959} - \frac{41\,(n-13)}{20170} + \frac{77\,(n-14)}{141371} - \frac{(n-15)}{6852} + \frac{(n-16)}{25572} - \frac{(n-17)}{95436} + \frac{(n-18)}{356172} - *\right]$$
……………………………… (9.10) * Values less than 1 x 10^{-6} an ignored

Note: {X} = 0 when X ≤ 0
{X } = 1 when X > 0

When the value of w is put at 2 and steps 1 and 2 repeated, each term in the above series was doubled, showing that the exponent of w in the series is 1. Likewise when l was put at 2, each term in the series was quadrupled showing that its exponent is 2, hence their inclusion in the series. Because equations (9) were generated for a 18-spanned beam not all the terms needs to be evaluated for beams with number of span n < 18. This explains the reason for the inclusion of the function {X}. Equations (9.1) to (9.10) represent the formulae for finding the internal support moments of any n-spanned uniformly loaded continuous beams of constant span, flexural rigidity and uniformly distributed load (UDL).

Relationship between M, w_1, w_2… w_n and n; keeping l constant

Recall that in equation (1) that M is dependent on all the span UDLs. When step 1 and 2 were carried out for the 1st interior support M_1 but this time putting $w_1 = 1$, $w_2 = w_3 = w_4 =$ … $= w_n = 0$ and l = 1 as shown in fig 2a, the series below was obtained.

$$M_1 = wl^2\left[\frac{(n-1)}{16} - \frac{(n-2)}{240} + \frac{(n-3)}{3360} - \frac{(n-4)}{46816} + \frac{(n-5)}{652080} - \frac{(n-6)}{9082320} + *\right]$$

If each term is expressed in terms of the corresponding term in equation (9.1) the equation becomes

$$M_1 = wl^2\left[\frac{\frac{1}{2}(n-1)}{8} - \frac{-\frac{1}{6}(n-2)}{40} + \frac{\frac{1}{24}(n-3)}{140} - \frac{-\frac{1}{88}(n-4)}{532} + \frac{\frac{1}{330}(n-5)}{1976} - \frac{-\frac{1}{1230}(n-6)}{7384} - *\right]$$ ……………… (10.1)

Where $\frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, \frac{1}{88}, \frac{1}{330}$ *and* $\frac{1}{1230}$ are the coefficients of w_1 in the first, second, third, fourth and fifth terms of the series. By repeating the same process for M_2, M_3, M_4… M_{10} the following series were developed.

$$M_2 = l^2\left[\frac{-\frac{1}{6}(n-2)}{10} - \frac{\frac{1}{24}(n-3)}{35} + \frac{-\frac{1}{88}(n-4)}{133} - \frac{\frac{1}{330}(n-5)}{494} + \frac{-\frac{1}{1230}(n-6)}{1846} - \frac{\frac{1}{4592}(n-7)}{6887} + \frac{-\frac{1}{17136}(n-8)}{25705} - *\right]$$
…………………. (10.2)

$$M_3 = l^2\left[\frac{\frac{1}{24}(n-3)3}{28} - \frac{-\frac{1}{88}(n-4)15}{532} + \frac{\frac{1}{330}(n-5)15}{1976} - \frac{-\frac{1}{1230}(n-6)15}{7384} + \frac{\frac{1}{4592}(n-7)15}{27548} - \frac{-\frac{1}{17136}(n-8)3}{20564} + \frac{\frac{1}{63954}(n-9)3}{76744} - *\right]$$
.... (10.3)

$$M_4 = l^2\left[\frac{-\frac{1}{88}(n-4)2}{19} - \frac{\frac{1}{330}(n-5)7}{247} + \frac{-\frac{1}{1230}(n-6)7}{923} - \frac{\frac{1}{4592}(n-7)14}{6887} + \frac{-\frac{1}{17136}(n-8)14}{25705} - \frac{\frac{1}{63954}(n-9)7}{47965} + \frac{\frac{1}{238678}(n-10)7}{179009} - *\right]$$
--------------- (10.4)

$$M_5 = l^2\left[\frac{\frac{1}{330}(n-5)11}{104} - \frac{-\frac{1}{1230}(n-6)209}{7384} + \frac{\frac{1}{4592}(n-7)47}{6195} - \frac{-\frac{1}{17136}(n-8)26}{12791} + \frac{\frac{1}{63954}(n-9)52}{95471} - \frac{\frac{1}{238678}(n-10)52}{356305} + - *\right]$$
.... (10.5)

$$M_6 = l^2\left[\frac{-\frac{1}{1230}(n-6)15}{142} - \frac{\frac{1}{4592}(n-7)195}{6887} + \frac{-\frac{1}{17136}(n-8)39}{5141} - \frac{\frac{1}{63954}(n-9)39}{19186} + \frac{\frac{1}{238678}(n-10)97}{178091} - \frac{\frac{1}{890760}(n-11)}{6852} + *\right]$$
.... (10.6)

$$M_7 = l^2\left[\frac{\frac{1}{4592}(n-7)41}{388} - \frac{-\frac{1}{17136}(n-8)165}{5828} + \frac{\frac{1}{63954}(n-9)93}{12259} - \frac{\frac{1}{238678}(n-10)21}{10331} + \frac{\frac{1}{890760}(n-11)73}{134027} - *\right]$$
.... (10.7)

$$M_8 = l^2\left[\frac{-\frac{1}{17136}(n-8)28}{265} - \frac{\frac{1}{63954}(n-9)128}{4521} + \frac{\frac{1}{238678}(n-10)11}{1450} - \frac{\frac{1}{890760}(n-11)41}{20170} + *\right]$$
.... (10.8)

$$M_9 = l^2\left[\frac{\frac{1}{63954}(n-9)153}{1448} - \frac{\frac{1}{238678}(n-10)287}{10137} + \frac{\frac{1}{890760}(n-11)302}{39809} - *\right]$$
.... (10.9)

$$M_{10} = l^2\left[\frac{\frac{1}{238678}(n-10)209}{1978} - \frac{\frac{1}{890760}(n-11)78}{2755} + *\right]$$
.... (10.10)

A careful study of equations (10) shows that the coefficient of w_1 for the term with {n – 1} is $\frac{1}{2}$. The coefficient of w_1 for the terms with {n – 2} is $-\frac{1}{6}$. The coefficent of w_1 for the terms with {n – 3} is $\frac{1}{24}$. The coefficient of w_1 for the terms with {n – 4} is $-\frac{1}{88}$. The coefficent of w_1 for the terms with {n – 5} is $\frac{1}{330}$. In like manner the coefficents of w_1 for the terms with {n – 6}, {n – 7}, {n – 8}, {n – 9}, {n – 10} and {n – 11} are $-\frac{1}{1230}, \frac{1}{4592}, -\frac{1}{17136}, \frac{1}{63954}, -\frac{1}{238678}$ *and* $\frac{1}{890760}$ respectively.
By putting $w_2 = 2$, $w_1 = w_3 = w_4 = w_5 = \ldots = w_n = 0$ and l =1 as in fig 2b and repeating the entire process used in generating equations (10), the coefficient of w_2 for the terms with {n –

1} was $\frac{1}{2}$. The coefficent of w_2 for the terms with {n – 2} was $-\frac{1}{2}$. The coefficent of w_2 for the terms with {n – 3} was– $\frac{1}{8}$. In like manner the coefficents of w_2 for the terms with {n – 4}, {n –5}, {n – 6}, {n –7}, {n – 8}, {n – 9}, {n – 10} and { n – 11} are $\frac{3}{88}, -\frac{1}{110}, \frac{1}{410}, -\frac{3}{4592}, \frac{1}{5712}, -\frac{1}{21318}, \frac{4}{318237}, and - \frac{1}{296921}$ respectively. Therefore the coefficients of w_1 and w_2 for any term with the same {X} are the same irrespective of the internal support moment being considered. Similar results were obtained for $w_3 = 1$, $w_1 = w_2 = w_4 = \ldots = w_n = 0$; $w_4 = 1$, $w_1 = w_2 = w_3 = w_5 = \ldots = w_n = 0$; $w_6 = 1$, $w_{i\neq 6} = 0$; w7 = 1, $w_{i\neq 7} = 0$ and so on. The values of these coefficients are shown in table 2.

Assuming the material of the beam is linear elastic and the deformation so small that it does not change the geometry of the beam appreciably then the law of superposition can be used in its analysis (Ghali et al, 1996). Let C_{ij} represent the coefficient of w_i on the j^{th} term i.e. the term with {n – j}. Then according to fig 2 any of the interior support moment is the sum of the corresponding interior support moments for $w_1 = 1, w_{i\neq 1} = 0$; $w_2 = 1, w_{i\neq 2} = 0$; $w_3 = 1, w_{i\neq 3} = 0$; $w_4 = 1, w_{i\neq 4} = 0$; ….; $w_n = 1, w_{i\neq n} = 0$. So that

$$M_1 = l^2\left[\frac{(c_{11}w_1 + c_{21}w_2 + \cdots + c_{n1}w_n)\langle n-1\rangle}{16} - \frac{(c_{12}w_1 + c_{22}w_2 + \cdots + c_{n2}w_n)\langle n-2\rangle}{240} + \frac{(c_{13}w_1 + c_{23}w_2 + \cdots + c_{n3}w_n)\langle n-3\rangle}{3360} - \frac{(c_{14}w_1 + c_{24}w_2 + \cdots + c_{n4}w_n)\langle n-4\rangle}{46816} + \frac{(c_{15}w_1 + c_{25}w_2 + \cdots + c_{n5}w_n)\langle n-5\rangle}{652080} - \frac{(c_{16}w_1 + c_{26}w_2 + \cdots + c_{n6}w_n)\langle n-6\rangle}{9082320} + *\right]$$

………………. (11)

The values of the coefficients: - c_{11}, c_{21},c_{31}… c_{n1} are the values under the {n – 1} column in table 2 and the non zero values are $\frac{1}{2}$ *and* $\frac{1}{2}$. In the same manner the coefficients on the second term: - c_{12}, c_{22}, c_{32} ,…, c_{n2} are the values under the {n - 2 } column and are $-\frac{1}{6}, \frac{1}{2}$ *and* $\frac{2}{3}$. Other coefficients can be picked from the table.

RESULT AND CONCLUSION

It follows that uniformly loaded continuous beams of equal spans can be analysed using an augmented form of equations 9 as stated below: -

$$M_1 = l^2\left[\frac{(n-1)W_{T1}}{8} - \frac{(n-2)W_{T2}}{40} + \frac{(n-3)W_{T3}}{140} - \frac{(n-4)W_{T4}}{532} + \frac{(n-5)W_{T5}}{1976} - \frac{(n-6)W_{T6}}{7384} + \frac{(n-7)W_{T7}}{27548} - \frac{(n-8)W_{T8}}{102820} + \frac{(n-9)W_{T9}}{383720} - *\right]$$

…………………. (12.1)

$$M_2 = l^2\left[\frac{(n-2)W_{T_2}}{10} - \frac{(n-3)W_{T_3}}{35} + \frac{(n-4)W_{T_4}}{133} - \frac{(n-5)W_{T_5}}{494} + \frac{(n-6W_{T_6})}{1846} - \frac{(n-7)W_{T_7}}{6887} + \frac{(n-8)W_{T_8}}{25705} - \frac{(n-9)W_{T_9}}{95930} + \frac{(n-10)W_{T_{10}}}{358018} - *\right]$$
.... (12.2)

$$M_3 = l^2\left[\frac{3(n-3)W_{T_3}}{28} - \frac{15(n-4)W_{T_4}}{532} + \frac{15(n-5)W_{T_5}}{1976} - \frac{15(n-6)W_{T_6}}{7384} + \frac{15(n-7)W_{T_7}}{27548} - \frac{3(n-8)W_{T_8}}{20564} + \frac{3(n-9)W_{T_9}}{76744} - \frac{2(n-10)W_{T_{10}}}{190943} + \frac{(n-11)W_{T_{11}}}{356304} - *\right]$$
... (12.3)

$$M_4 = l^2\left[\frac{2(n-4)W_{T_4}}{19} - \frac{7(n-5)W_{T_5}}{247} + \frac{7(n-6)W_{T_6}}{923} - \frac{14(n-7)W_{T_7}}{6887} + \frac{14(n-8)W_{T_8}}{25705} - \frac{7(n-9)W_{T_9}}{47965} + \frac{7(n-10)W_{T_{10}}}{179009} - \frac{2(n-11)W_{T_{11}}}{190877} + \frac{2(n-12)W_{T_{12}}}{712363} - *\right]$$
.... (12.4)

$$M_5 = l^2\left[\frac{11(n-5)W_{T_5}}{104} - \frac{209(n-6)W_{T_6}}{7384} + \frac{47(n-7)W_{T_7}}{6195} - \frac{26(n-8)W_{T_8}}{12791} + \frac{52(n-9)W_{T_9}}{95471} - \frac{52(n-10)W_{T_{10}}}{356305} + \frac{26(n-11)W_{T_{11}}}{664873} - \frac{5(n-12)W_{T_{12}}}{477181} + \frac{(n-13)W_{T_{13}}}{356173} - *\right]$$
.... (12.5)

$$M_6 = l^2\left[\frac{15(n-6)W_{T_6}}{142} - \frac{195(n-7)W_{T_7}}{6887} + \frac{39(n-8)W_{T_8}}{5141} - \frac{39(n-9)W_{T_9}}{19186} + \frac{97(n-10)W_{T_{10}}}{178091} - \frac{(n-11)W_{T_{11}}}{6852} + \frac{(n-12)W_{T_{12}}}{25572} - \frac{(n-13)W_{T_{13}}}{95436} + \frac{(n-14)W_{T_{14}}}{356172} - *\right]$$
.. (12.6)

$$M_7 = l^2\left[\frac{41(n-7)W_{T_7}}{388} - \frac{165(n-8)W_{T_8}}{5828} + \frac{93(n-9)W_{T_9}}{12259} - \frac{21(n-10)W_{T_{10}}}{10331} + \frac{73(n-11)W_{T_{11}}}{134027} - \frac{(n-12)W_{T_{12}}}{6852} + \frac{(n-13)W_{T_{13}}}{25572} - \frac{(n-14)W_{T_{14}}}{95436} + \frac{(n-15)W_{T_{15}}}{356172} - *\right]$$
... (12.7)

$$M_8 = l^2\left[\frac{28(n-8)W_{T_8}}{265} - \frac{128(n-9)W_{T_9}}{4521} + \frac{11(n-10)W_{T_{10}}}{1450} - \frac{41(n-11)W_{T_{11}}}{20170} + \frac{78(n-12)W_{T_{12}}}{143207} - \frac{(n-13)W_{T_{13}}}{6852} + \frac{(n-14)W_{T_{14}}}{25572} - \frac{(n-15)W_{T_{15}}}{95436} + \frac{(n-16)W_{T_{16}}}{356172} - *\right]$$
.... (12.8)

$$M_9 = l^2\left[\frac{153(n-9)W_{T_9}}{1448} - \frac{287(n-10)W_{T_{10}}}{10137} + \frac{302(n-11)W_{T_{11}}}{39809} - \frac{41(n-12)W_{T_{12}}}{20170} + \frac{76(n-13)W_{T_{13}}}{139535} - \frac{(n-14)W_{T_{14}}}{6852} + \frac{(n-15)W_{T_{15}}}{25572} - \frac{(n-16)W_{T_{16}}}{95436} + \frac{(n-17)W_{T_{17}}}{356172} - *\right]$$
.... (12.9)

$M_{10} = l^2 \Big[\frac{209(n-10)W_{T10}}{1978} - \frac{78(n-11)W_{T11}}{2755} + \frac{379(n-12)W_{T12}}{49959} - \frac{41(n-13)W_{T13}}{20170} + \frac{77(n-14)W_{T14}}{141371} - \frac{(n-15)W_{T15}}{6852} + \frac{(n-16)W_{T16}}{25572} - \frac{(n-17)W_{T17}}{95436} + \frac{(n-18)W_{T18}}{356172} - *\Big]$
.... (12.10)

*Wre

$W_{T1} = \frac{1}{2}w_1 + \frac{1}{2}$.............................. (13.1)

$W_{T2} = -\frac{1}{6}w_1 + \frac{1}{2}w_2 + \frac{2}{3}w_3$.................. (13.2)

$W_{T3} = \frac{1}{24}w_1 - \frac{1}{8}w_2 + \frac{11}{24}w_3 + \frac{5}{8}w_4$.......... (13.3)

$W_{T4} = -\frac{1}{88}w_1 + \frac{3}{88}w_2 - \frac{1}{8}w_3 + \frac{41}{88}w_4 + \frac{7}{11}w_5$ .... (13.4)

$W_{T5} = \frac{1}{330}w_1 - \frac{1}{110}w_2 + \frac{1}{30}w_3 - \frac{41}{330}w_4 + \frac{51}{110}w_5 + \frac{19}{30}w_6$ .... (13.5)

$W_{T6} = -\frac{1}{1230}w_1 + \frac{1}{410}w_2 - \frac{11}{1230}w_3 + \frac{1}{30}w_4 - \frac{51}{410}w_5 + \frac{571}{1230}w_6 + \frac{26}{41}w_7$ .... (13.6)

$W_{T7} = \frac{1}{4592}w_1 - \frac{3}{4592}w_2 + \frac{11}{4592}w_3 - \frac{1}{112}w_4 + \frac{153}{4592}w_5 - \frac{571}{4592}w_6 + \frac{381}{821}w_7 + \frac{71}{112}w_8$ .... (13.7)

$w_{T8} = -\frac{1}{17136}w_1 + \frac{1}{5712}w_2 - \frac{11}{17136}w_3 + \frac{41}{17136}w_4 - \frac{1}{112}w_5 + \frac{95}{2851}w_6 - \frac{339}{2726}w_7 + \frac{569}{1226}w_8 + \frac{97}{153}w_9$
.... (13.8)

$W_{T9} = \frac{1}{63954}w_1 - \frac{1}{21318}w_2 + \frac{1}{5814}w_3 - \frac{41}{63954}w_4 + \frac{1}{418}w_5 - \frac{286}{32033}w_6 + \frac{89}{2671}w_7 - \frac{241}{1938}w_8 + \frac{711}{1532}w_9 + \frac{265}{418}w_{10}$
.... (13.9)

$W_{T10} = -\frac{1}{238678}w_1 + \frac{4}{318237}w_2 - \frac{1}{21698}w_3 + \frac{12}{69857}w_4 - \frac{77}{120119}w_5 + \frac{1}{418}w_6 - \frac{355}{39761}w_7 + \frac{271}{8133}w_8 - \frac{555}{4463}w_9 + \frac{181}{390}w_{10} + \frac{362}{571}w_{11}$
.... (13.10)

$W_{T11} = \frac{1}{890760}w_1 - \frac{1}{296921}w_2 + \frac{5}{404891}w_3 - \frac{7}{152081}w_4 + \frac{25}{145549}w_5 - \frac{1}{1560}w_6 + \frac{1066}{445589}w_7 - \frac{331}{37073}w_8 + \frac{90}{2701}w_9 - \frac{193}{1552}w_{10} + \frac{989}{2131}w_{11} + \frac{989}{1560}w_{12}$
.... (13.11)

$W_{T12} = -\frac{-2}{604429}w_3 + \frac{1}{81082}w_4 - \frac{6}{130367}w_5 + \frac{1}{5822}w_6 - \frac{1}{1560}w_7 + \frac{1326}{554269}w_8 - \frac{337}{37745}w_9 + \frac{90}{2701}w_{10} - \frac{193}{1552}w_{11} + \frac{1351}{2911}w_{12} + \frac{1351}{2131}w_{13}$
.. (13.12)

$W_{T13} = -\frac{3}{907814}w_4 + \frac{3}{243268}w_5 - \frac{1}{21728}w_6 + \frac{1}{5822}w_7 - \frac{1}{1560}w_8 + \frac{1237}{517067}w_9 - \frac{336}{37633}w_{10} + \frac{90}{2701}w_{11} - \frac{193}{1552}w_{12} + \frac{1170}{2521}w_{13} + \frac{1351}{2131}w_{14}$
.. (13.13)

$$W_{T14} = -\frac{1}{302628}w_5 + \frac{1}{81090}w_6 - \frac{1}{21728}w_7 + \frac{1}{5822}w_8 - \frac{1}{1560}w_9 + \frac{1260}{526681}w_{10} - \frac{336}{37633}w_{11} + \frac{90}{2701}w_{12} - \frac{193}{1552}w_{13} + \frac{1170}{2521}w_{14} - \frac{1351}{2131}w_{15}$$
.... (13.14)

$$W_{T15} = -\frac{1}{302622}w_6 + \frac{1}{81090}w_7 - \frac{1}{21728}w_8 + \frac{1}{5822}w_9 - \frac{1}{1560}w_{10} + \frac{1253}{523755}w_{11} - \frac{336}{37633}w_{12} + \frac{90}{2701}w_{13} - \frac{193}{1552}w_{14} + \frac{1170}{2521}w_{15}$$
.... (13.15)

$$W_{T16} = -\frac{1}{302622}w_7 + \frac{4}{324361}w_8 - \frac{1}{21728}w_9 + \frac{1}{5822}w_{10} - \frac{1}{1560}w_{11} + \frac{1231}{514559}w_{12} - \frac{336}{37633}w_{13} + \frac{90}{2701}w_{14} - \frac{193}{1552}w_{15}$$
.... (13.16)

$$W_{T17} = -\frac{1}{302627}w_8 + \frac{1}{81090}w_9 - \frac{1}{21728}w_{10} + \frac{1}{5822}w_{11} - \frac{1}{1560}w_{12} + \frac{1254}{524173}w_{13} - \frac{336}{37633}w_{14} + \frac{90}{2701}w_{15}$$
.... (13.17)

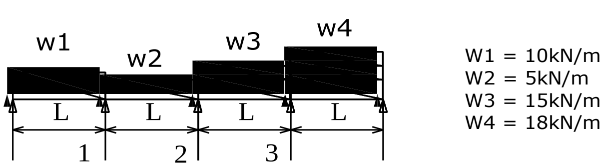
$$W_{T18} = -\frac{1}{302626}w_9 + \frac{1}{81090}w_{10} - \frac{1}{21728}w_{11} + \frac{1}{5822}w_{12} - \frac{1}{1560}w_{13} + \frac{1254}{524173}w_{14} - \frac{336}{37633}w_{15}$$
.... (13.18)

With equations 12 above the first internal support of moments of an 18 spanned continuous beam can be obtained. For continuous beams with number of spans n < 18, it is not necessary to evaluate all the terms in the series and the function {X} helps to prevent such possible mistakes.

A comparison of the values generated by these finite series with the internal moment coefficient of the Reinforced Concrete Designer's Handbook is presented in table 3 in Appendix E

Below is a practical demonstration of the use of the series in the analyses of a uniformly loaded continuous beam with 4 spans:-

Numerical Example 1



From equations (13.1) to (13.3)

$$w_{T1} = \frac{1}{2} \times 10 + \frac{1}{2} \times 5 = 7.5$$

$$w_{T2} = -\frac{1}{6} \times 10 + \frac{1}{2} \times 5 + \frac{2}{3} \times 15 = 18.8333$$

$$W_{T3} = \frac{1}{24} \times 10 - \frac{1}{8} \times 5 + \frac{11}{24} \times 15 + \frac{5}{8} \times 18 = 17.9167$$

From equations (12.1) to (12.3)

$$M_1 = l^2 \left[\frac{7.5}{8} - \frac{10.8333}{40} + \frac{17.9167}{140}\right] = 0.7946 l^2$$

$$M_2 = l^2 \left[\frac{10.8333}{10} - \frac{17.9167}{35}\right] = 0.5714 l^2$$

$$M_3 = l^2 \left[\frac{3 \times 17.9167}{28}\right] = 1.9196 l^2$$

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APPENDIX A

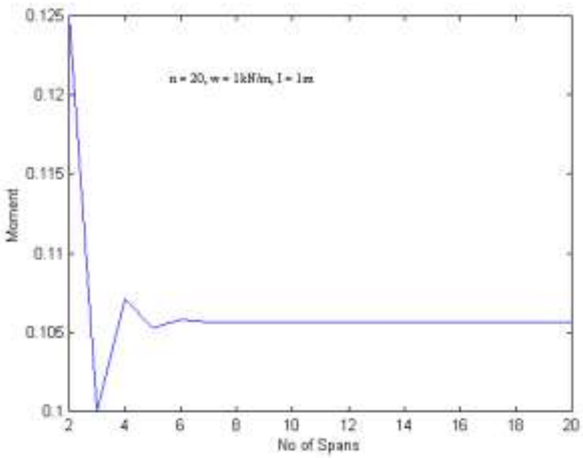


Figure 1: A Graph of the first internal support M against number of spans n

APPENDIX B

Table 1: Results of Internal support moment coefficients (w & l constant)

No of Spans = 18 **w = 1kN/m** **l = 1m**

n	2	3	4	5	6	7	8	9	10
M1	1/8	1/10	3/28	2/19	11/104	15/142	41/388	28/265	153/1448
M2	-	1/10	1/14	3/38	1/13	11/142	15/194	41/530	14/181
M3	-	-	3/28	3/38	9/104	6/71	33/388	9/108	123/1443
M4	-	-	-	2/19	1/13	6/71	8/97	22/265	15/181
M5	-	-	-	-	11/104	11/142	33/388	22/265	121/1448
M6	-	-	-	-	-	15/142	15/194	9/106	15/181
M7	-	-	-	-	-	-	41/388	41/530	123/1448
M8	-	-	-	-	-	-	-	28/265	14/181
M9	-	-	-	-	-	-	-	-	153/1448

n	11	12	13	14	15	16	17	18
M1	209/1978	571/5404	390/3691	390/3691	390/3691	390/3691	390/3691	390/3691
M2	153/1978	209/2702	195/2521	195/2521	195/2521	195/2521	195/2521	195/2521
M3	84/989	459/5404	340/4003	234/2755	234/2755	234/2755	234/2755	234/2755
M4	82/989	16/193	306/3691	209/2521	193/2328	193/2328	193/2328	193/2328
M5	165/1978	451/5404	308/3691	541/6483	121/1450	302/3619	423/5069	302/3619
M6	165/1978	225/2702	307/3685	210/2521	229/2749	224/2689	225/2701	225/2701
M7	82/989	451/5404	307/3685	420/5039	1148/13775	784/9407	857/10283	836/10031
M8	84/989	16/193	308/3691	210/2521	1148/13775	1568/18817	4284/51409	2926/35113
M9	153/1978	459/5404	306/3691	541/6483	230/2761	784/9407	4284/51409	5852/70223
M10	209/1978	209/2702	340/4003	209/2521	121/1450	224/2689	857/10283	2926/35113
M11	-	571/5404	195/2421	234/2755	193/2328	302/3619	225/2701	836/10031
M12	-	-	390/3691	195/2521	234/2755	193/2328	423/5069	225/2701
M13	-	-	-	390/3691	195/2521	234/2755	193/2328	302/3619
M14	-	-	-	-	390/3691	195/2521	234/2755	193/2328
M15	-	-	-	-	-	390/3691	195/2521	234/2755
M16	-	-	-	-	-	-	390/3691	195/2521
M17	-	-	-	-	-	-	-	390/3691
M18	-	-	-	-	-	-	-	-
M19	-	-	-	-	-	-	-	-

APPENDIX C

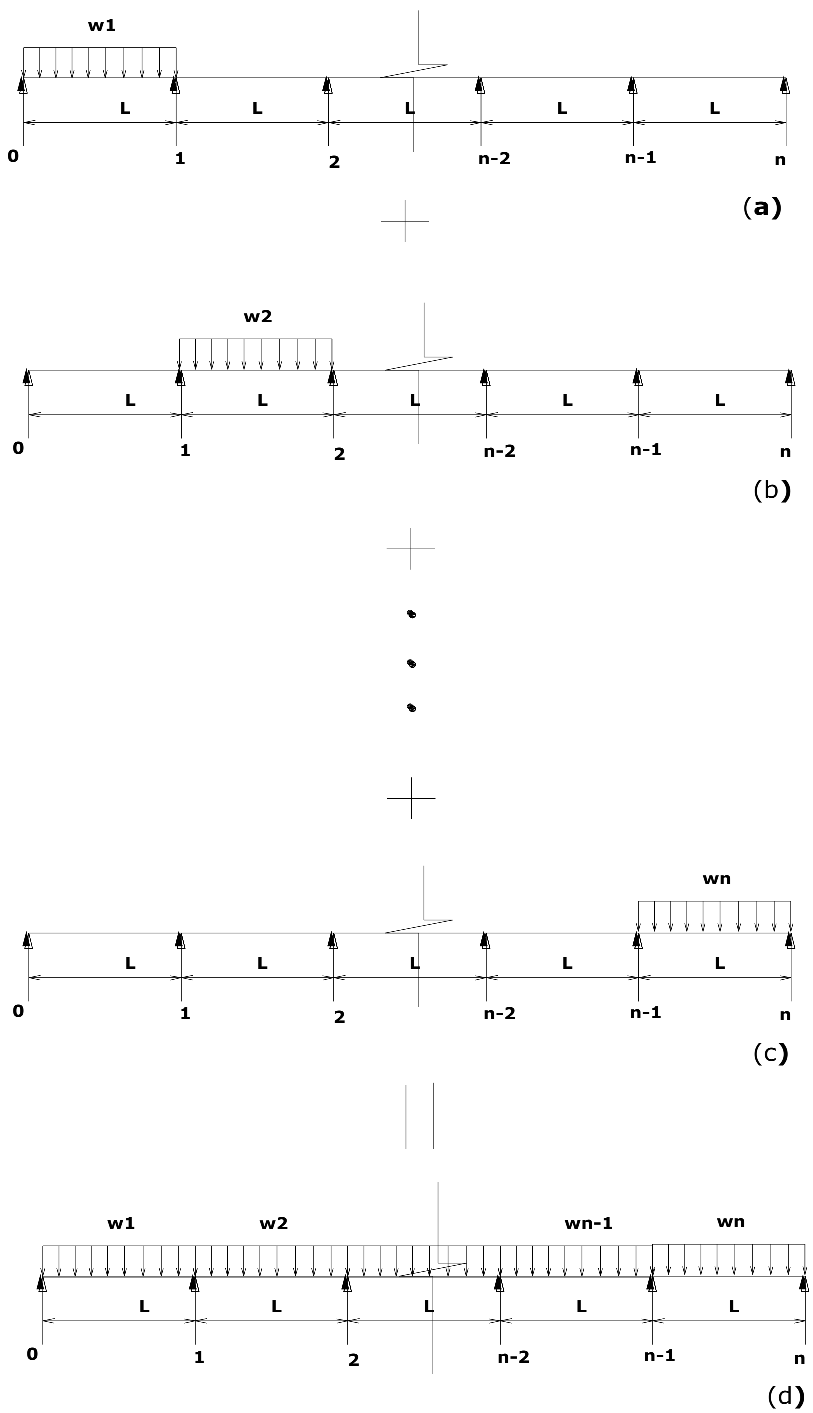


Figure 2: Method of superposition in uniformly loaded continuous beams

APPENDIX D

Table 2: Coefficients of the uniformly distributed load w on each term of the Finite series

c_{ij}	< n - 1 >	< n - 2 >	< n - 3 >	< n - 4 >	< n - 5 >	< n - 6 >	< n - 7 >	< n - 8 >	< n - 9 >	< n
w_1	1/2	-1/6	1/24	-1/88	1/330	-1/1230	1/4592	-1/17136	1/63954	-1/2
w_2	1/2	½	-1/8	3/88	-1/110	1/410	-3/4592	1/5712	-1/21318	4/3
w_3	-	2/3	11/24	-1/8	1/30	-11/1230	11/4592	-11/17136	1/5814	-1/2
w_4	-	-	5/8	41/88	-41/330	1/30	-1/112	41/17136	-41/63954	12/
w_5	-	-	-	7/11	51/110	-51/410	153/4592	-1/112	1/418	-77/
w_6	-	-	-	-	19/30	571/1230	-571/4592	95/2851	-286/32033	1/4
w_7	-	-	-	-	-	26/41	381/821	-339/2726	89/2671	-35
w_8	-	-	-	-	-	-	71/112	569/1226	-241/1938	271
w_9	-	-	-	-	-	-	-	97/153	711/1532	-55
w_{10}	-	-	-	-	-	-	-	-	265/418	181
w_{11}	-	-	-	-	-	-	-	-	-	362
w_{12}	-	-	-	-	-	-	-	-	-	-
w_{13}	-	-	-	-	-	-	-	-	-	-
w_{14}	-	-	-	-	-	-	-	-	-	-
w_{15}	-	-	-	-	-	-	-	-	-	-

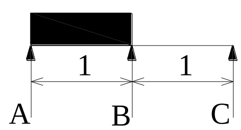
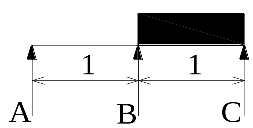
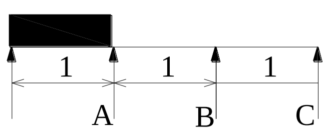
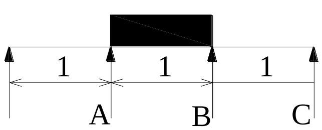
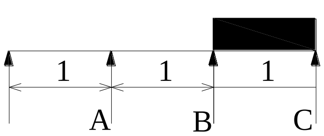
c_{ij}	< n - 11 >	< n - 12 >	< n - 13 >	< n - 14 >	< n - 15 >	< n - 16 >	< n - 17 >	< n
w_1	1/890760	*						
w_2	-1/296921	*						
w_3	5/404891	-2/604429	*					
w_4	-7/152081	1/81082	-3/907814	*				
w_5	25/145549	-6/130367	3/243268	-1/302628	*			
w_6	-1/1560	1/5822	-1/21728	1/81090	-1/302632	*		
w_7	1066/445589	-1/1560	1/5822	-1/21728	1/81090	-1/302632	*	
w_8	-331/37073	1326/554269	-1/1560	1/5822	-1/21728	4/324361	-1/302627	*
w_9	90/2701	-337/37745	1237/517067	-1/1560	1/5822	-1/21728	1/81090	-1/30
w_{10}	-193/1552	90/2701	-336/37633	1260/526681	-1/1560	1/5822	-1/21728	1/81
w_{11}	989/2131	-193/1552	90/2701	-336/37633	1253/523755	-1/1560	1/5822	-1/2
w_{12}	989/1560	1351/2911	-193/1552	90/2701	-336/37633	1254/524173	-1/1560	1/58
w_{13}	-	1351/2131	1351/2911	-193/1552	90/2701	-336/37633	1231/514559	-1/15
w_{14}	-	-	1351/2131	1351/2911	-193/1552	90/2701	-336/37633	1231
w_{15}	-	-	-	1351/2131	1351/2911	-193/1552	90/2701	-336

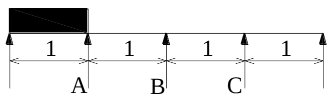
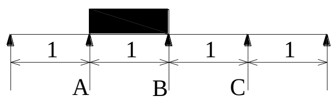
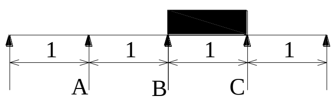
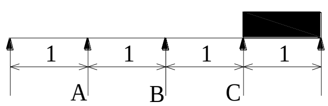
* Value is less than 1×10^{-6} and can be ignored

APPENDIX E

Table 3:- Comparison of the Support moment coefficients of Table 43 of Reinforced Concrete Designer's Manual with the Finite series solution.

Q = Support Moment Coefficient, Span l = 1m, uniformly distributed load w = 1kN/m

No of Spans	Loaded Beams	Support Moment Coefficients	Table 43 Reinforced Concrete Designer's Handbook	Finite Series, Equations (12.1 - 12.10)
2		QA QB QC	- 0.0625 -	- 0.0625 -
		QA QB QC	- 0.0625 -	- 0.0625 -
	Both Spans loaded with identical loads	QB	0.125	0.125
3		QA QB QC	0.0667 -0.0167 -	0.06667 -0.01667 -
		QA QB QC	0.05 0.05 -	0.05 0.05 -
		QA QB QC	-0.0167 0.0667 -	-0.01667 0.06667

	All Spans loaded with the identical load	QA = QB	0.1	0.1
4		QA QB QC	0.067 -0.0179 0.0045	0.06696 -0.01786 0.00446
		QA QB QC	0.0491 0.0536 -0.0134	0.04911 0.05357 -0.01339
		QA QB QC	-0.0134 0.0536 0.0491	-0.01339 0.05357 0.04911
		QA QB QC	0.0045 -0.0179 0.067	0.00446 -0.01786 0.06696
	All Spans loaded with the identical load	QA QB QC	0.1071 0.0714 0.1071	0.10714 0.07143 0.10714

Note: - Hogging moments were treated as positive and sagging moments negative.