Journal of Physical Science and Innovation
ISSN: 2277-0119
Volume 9, No. 3, 2017

GLOBAL STABILITY AND BIFURCATION ANALYSIS OF A MATHEMATICAL MODEL OF DIABETES

Modu G.U.[1], Askira S.A., Ahmed I. & Mohammed M.
Department of Statistics,
Ramat Polytechnic, Maiduguri, Borno State, Nigeria.
Email: goni_umar@yahoo.com

ABSTRACT

Global stability and bifurcation analysis of an existing mathematical model for monitoring diabetic population is treated in this paper. The work is an extension of the work in [2] and [3]. The equilibrium point in the linear case of the model is seen to be globally asymptotically stable. The system, in the nonlinear case, is seen to undergo bifurcation (trans-critical and Hopf bifurcation).

Keywords: Equilibrium Point, Global Stability, Bifurcation.

INTRODUCTION

Global stability and bifurcation analysis of systems are important scientifically. They provide us with the long term behavior of a system and the transitions and instabilities as some control parameter is varied respectively [7]. The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. In this work, following the model proposed in [2] and [3], we intend to look into the qualitative nature of the model analytically. This is because most of the work were done numerically and thus the need for analytical results and investigating other qualitative nature of the system shall give us more insight into it the long term behavior. The aim of this work therefore, is to provide a short review of the existing model in [2] and [3] to provide the global and bifurcation analysis of the equilibria of the system in the linear [2] and the nonlinear [3] cases respectively. To achieve this, we employ the use of

Lyapunov function to analyze the global stability of the equilibrium of the linear case and direct check (analytically) to see weather the equilibria in the nonlinear case undergo bifurcation. This paper is organized in the following order: In the first section we recall some of the results in [2] and [3] as regard the stability of the equilibria. The second section gives the global stability of the equilibrium point in the linear case of the system. The third section shall give the bifurcation analysis of the equilibria in the nonlinear case.

THE MODEL

Linear Case

For the fact that diabetes is not a communicable disease, there were no issues of reproduction number in this work. The linear case of the model is given as follows:

$$C' = -(\lambda+\theta)C + \lambda N,$$
$$N' = I-(\nu+\delta)C-\mu N, \qquad (1)$$

$\theta = \gamma+\mu+\nu+\delta$, $N = N(t)$, $C = C(t)$, $I = I(t)$, $N(0) = N_0$, $C(0) = C_0$.

The linear system has unique equilibrium point given by:

$E_1 = \left(\frac{\lambda I^*}{\mu\lambda+\mu\theta+\lambda\delta+\lambda\nu}, \frac{(\lambda+\theta)I^*}{\mu\lambda+\mu\theta+\lambda\delta+\lambda\nu}\right)$, and was found to be locally stable.

Non Linear Case

In the linear case, λ, the rate of developing complications of the disease, was considered
to be constant. In this case, λ is defined as:

$$\lambda = \beta\frac{C}{N}, \quad 0 < \beta \leq 1.$$

Thus, the system become nonlinear and may consequently be written in the form:

$$C' = (\beta-\theta)C-\beta\frac{C^2}{N},$$

$$N' = I-(\nu+\delta)C-\mu N. \qquad (2)$$

The equilibrium points of this nonlinear system are:

the trivial equilibrium point

$E_2 = (0, \frac{I^*}{\mu})$,

and the nontrivial equilibrium point

$E_3 = (\frac{(\beta-\theta)I^*}{\mu\beta+(\nu+\delta)(\beta-\theta)}, \frac{(\beta-\theta)I^*}{\mu\beta+(\nu+\delta)(\beta-\theta)})$, $\beta-\theta > 0$.

The trivial fixed point was found to be unstable while the nontrivial fixed point was seen to conditionally stable. However, checks using Routh-Hurwitz criterion shows that the nontrivial equilibrium point was seen to be stable.

Existence and Positivity of Solution

Since the model (1) describes human population it is necessary to show that the state Variables C and N are nonnegative for all $t \geq 0$. Solutions with positive initial data remain positive for all $t \geq 0$ and are bounded. Based on biological consideration therefore, the model (1) will be studied in the region

$$\Omega = \{ (C,N) \in \mathbb{R}^2_+ \; 2+ : C \geq 0, N \leq I/\mu \}.$$

Theorem 1.1.

Let $C(t) \geq 0, N(t) > 0$. The solution C, N to the model (1) exists, is unique and remains in Ω for all $t \geq 0$.

Proof

The right-hand side of the system (1) is continuous with partial derivatives in Ω [8]. We use the method of contradiction as in [1] to show that Ω is positively invariant. Under the initial conditions, assume that there exists a time t1 such that C(t1) = 0, C '(t_1) < 0,

N(t_1) > 0 for 0 < t < t_1,

In this first case (t1):

$$C'(t_1) = \lambda N,$$
$$> 0,$$

which is a contradiction. Meaning C(t) > 0.

Thus, C remains nonnegative.

Also, since N(t) ≥ C(t), then

N'= N '(t),

$$= \frac{dN}{dt},$$

$$\Rightarrow \frac{dN}{dt} = I - (\nu + \delta)C - \mu N,$$

$$\leq I - \mu N,$$

$$\Rightarrow \frac{dN}{dt} + \mu N \leq I. \qquad (3)$$

To solve for N in (3), we apply integrating factor method. Thus, solution to (3) is given as follows:

$$N \leq \frac{I}{\mu} + \tilde{d}e^{-\mu t},$$

At t = 0,

$$N_0 \leq \frac{I}{\mu} + \tilde{d},$$
$$\Longrightarrow \tilde{d} \geq N_0 - \frac{I}{\mu},$$
$$\Longrightarrow N \leq \frac{I}{\mu} + e^{-\mu t}(N_0 - \frac{I}{\mu}),$$
$$N = \frac{I}{\mu}(1 - e^{-\mu t}) + N_0 e^{-\mu t},$$

As $t \to \infty$,

$$N \leq \frac{I}{\mu},$$

which implies that N(t) is bounded and a unique solution exists for all $t \geq 0$ in Ω. This completes the proof. We conclude that the model is epidemiologically well-posed. It should be noted that the region of feasibility in both linear and nonlinear cases are the same.

Global Stability Analysis

The goal of this section is to establish sufficient condition on the global asymptotic stability of the equilibrium point in the linear case of the system. The result is obtained by employing the use of Lyapunov function considered in [5] with little modification.

Global Stability Analysis of E_1

Theorem 2.1

The fixed point E1 of the linear system is globally asymptotically stable if $\lambda < \frac{\mu(\lambda+\theta)}{I}$.

Proof

Consider the Lyapunov function

$$F = (C - C^*) + \frac{(\lambda+\theta)}{I}(N - N^*), \qquad (4)$$

with derivative along the solution curve:

$$F' = C' + \frac{(\lambda+\theta)}{I}N',$$

$$= [-(\lambda+\theta)C + \lambda N] + \frac{(\lambda+\theta)}{I}[I - (\nu+\delta)C - \mu N],$$

$$= -(\lambda+\theta)C + \lambda N + (\lambda+\theta) - \frac{(\lambda+\theta)(\nu+\delta)}{I}C - \frac{\mu(\lambda+\theta)}{I}N,$$

$$= -\left[(\lambda+\theta) + \frac{(\lambda+\theta)(\nu+\delta)}{I}\right]C + \left[\lambda - \frac{\mu(\lambda+\theta)}{I}\right]N + (\lambda+\theta),$$

< 0, provided that $\lambda < \frac{\mu(\lambda+\theta)}{I}$,

Thus, $F' = 0$ only if $C = C^*$ and $N = N^*$. This indicates that the largest invariant set in $\{(C,N) \in \Omega : F = 0\}$ is the singleton E_1. Therefore by LaSalle's invariance principle [6], E_1 is globally asymptotically stable in Ω. This result shows that the disease establishes itself over certain period of time.

Trans-critical and Hopf Bifurcations of the Nonlinear Case

Consider $\eta = \beta - \theta$ to be the bifurcation parameter.

(a) Bifurcation at E_2

The characteristic polynomial here is given by:

$$p_1(\chi) = \chi^2 + [\mu - (\beta - \theta)]\chi - \mu(\beta - \theta),$$

whose zeros are: $\chi_1 = -\mu < 0$ and $\chi_2 = \beta - \theta > 0$.

Theorem3.1. [7]

If $\eta = 0$, the equilibrium point E_2 undergoes trans-critical bifurcation that generates the unstable equilibrium point E_2. For $\eta < 0$, there is a stable equilibrium at E_2 and an unstable equilibrium point at E_3. As η increases, the unstable equilibrium point E_2 approaches and coalesces with E_3 when $\eta = 0$ (see Section 1), but still unstable. Finally, when $\eta > 0$, the equilibrium point E_2 become unstable and E_3 is now stable (as seen above). Thus, the exchange of stabilities has taken place between the two equilibrium points. It should be noted that the equilibrium point E_2 cannot undergo Hopf bifurcation since none of the roots of the characteristic equation can be purely imaginary.

(b) Bifurcation at E_3

Here, we observe that the characteristic polynomial is:

$$p_2(\chi) = \chi^2 + (\beta - \theta + \mu)\chi + \mu(\beta - \theta) + \frac{(\nu + \delta)(\beta - \theta)^2}{\beta},$$

Since $\frac{(\nu+\delta)(\beta-\theta)^2}{\beta} > 0$, the system does not undergo trans-critical bifurcation at E_3. Hence, we study the Hopf bifurcation. Clearly, if $\eta = \mu$, the system undergoes Hopf bifurcation at E_3. In this case, the roots of the polynomial are $\chi_{1,2} = \pm i\sqrt{\omega}$, $\omega = \mu(\beta - \theta) + \frac{(\nu+\delta)(\beta-\theta)^2}{\beta}$

CONCLUSION

In this paper we investigated the behavior of a mathematical model for monitoring Diabetic population. We have seen that the linear case of the model is globally asymptotically stable. The nonlinear system displays a trans-critical bifurcation at E_2 and at the other equilibrium point, E_3, a Hopf bifurcation.

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Reference to this paper should be made as follows: Modu G.U., et al (2017), Global Stability and Bifurcation Analysis of a Mathematical Model of Diabetes. J. of Physical Science and Innovation, Vol. 9, No. 3, Pp. 56-63.