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# ON LORENTZIAN SYSTEM OF DIFFERENTIAL EQUATION

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## ABSTRACT

This paper proposes and demonstrates a new integration of the theory of buttery effect of chaos theory in relation to initial conditions. It is very unusual for a mathematical idea to disseminate into the society at large. An interesting example is chaos theory, popularized by Lorenz's butterfly effect: "does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" A tiny cause can generate big consequences. We have suggested an analytic approach to Lorentz system and with the use of MACTCONT, toolboxes in MATLAB, we have computed the system using different initial conditions and tracked the behaviuor of the system, eventually ending up by showing impossibility of weather forecasting being ineffective after one week and become chaotic.

**Keywords: Homtopy Continuation Method, Butterfly Effect, MATCONT.**

## INTRODUCTION

In common usage, chaos simply means a state of disorder [7]. However, in the theory of chaos, the term is defined more precisely. Although there is no universally accepted mathematical definition of chaos, a commonly used definition says that, for a dynamical system to be classified as chaotic, it must have the following properties, [5].

- Be sensitive to initial conditions;
- Be topologically mixing; and
- Have denseperiodic Orbits.

Sensitivity to initial conditions means that each point in a chaotic system is arbitrarily closely approximated by other points with significantly different future paths, or trajectories. Thus, an arbitrarily small change, or perturbation, of the current trajectory may lead to significantly different future behavior. It has been shown that in some cases the last two properties in the above actually imply sensitivity to initial conditions,[3] and [1], and if attention is restricted to intervals, the second property implies the other two[12]. (An alternative, and in general weaker, definition of chaos uses only the first two properties in the above list)[8]. It is interesting that the most practically significant property, that of sensitivity to initial conditions, is redundant in the definition, being implied by two (or for intervals, one) purely topological properties, which are therefore of greater interest to mathematicians. The Lorenz system of differential equations arose from the work of a meteorologist/mathematician Edward N. Lorenz. As he was computing numerical solutions of the system of three differential equations that he came up with, he noticed that initial conditions with small differential eventually produced vastly different solutions. What he had observed was sensitivity to initial condition, a characteristic of chaos. His observation led him to further study of the system, and since that time, about 1963, the Lorenz system became one of the widely studied systems of ordinary differential equation (ODE) because of its wide range of behaviour.

The system of differential equation Lorenz used was

$$\left.\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -rx - y + xz \\ \dot{z} &= -\beta z + xy \end{aligned}\right\} \tag{1}$$

where $\sigma$, r, and $\beta$ are positive parameters which denotes physical characteristics of air flow. The variable x corresponds to the amplitude of convective currents in the air cells, y to the falling currents and z, to the deviation of the temperature from the normal temperature in the cell. Even though a definition of chaos has not been agreed upon by mathematician, two properties that are generally agreed to characterize it are sensitivity to initial conditions and the presence of period – doubling cycles leading to chaos. The Lorenz system exhibits both of these characteristic. We already mentioned the first and the second simply the presence of limit cycles which repeatedly double their period as r is varied in one direction unit the orbits begin to wander chaotically. We will explore these dynamics and other behaviour of the Lorenz system.

## RELATED WORK ON CHAOTIC SYSTEMS

Reference [2] published "The Fractal Geometry of Nature" which became a classical of chaos theory. Biological systems such as the branching of the circulatory and bronchial system proved to fit a fractal model. Reference [6] published Chaos Making a New Science, which became a best-seller and introduced the general principal of chaos theory as well as its history to the broad public, through Gleick history under emphasized important sovient contribution. Initially the domain of a few, isolated individuals, chaos theory progressively emerged as a trans-disciplinary and institutional discipline, mainly under the name of non-linear system analysis. Alluding to Thomas Kuhn's concept of paradigm shift exposed in the structure of scientific revolutions (1962) many "chaologists" (as some described themselves) claimed that this new theory was an example of such a shift, a thesis upheld by Gleick. The availability of cheaper, more powerful computers broaden the applicability of chaos theory, currently chaos theory continues to be a very active area of research involving many different disciplines (mathematics, topology, physics, social system, population modeling, biology, meteorology, astrophysics, information theory, computation neuroscience)

## METHODS

This section discusses the methodology which involves first order Differential equation such as separable differential equation, homogeneous equations, linear equation, Bernoulli equations and exact differential equation. We also include the idea of algebraic topology, the homotopy principle coupled with numerical analysis method, numerical continuation in achieving our result.

### Homotopy

If f(x) is the system of nonlinear equations to be solved and g(x) is a second simpler system of the same number of equations, the homotopy function might be constructed as

$$H(x, t) = t\, f(x) + (1 - t)\, g(x) = 0 \quad 0 \leq t \leq 1,\ x \in R^n$$

where t is a scalar homotopy parameter which is gradually varied from 0 to 1 as the path is tracked from starting point to a solution.

### Dynamical System

Consider the dynamical system

$$\left.\begin{aligned}\dot{x} &= -10x + 10y \\ \dot{y} &= -rx - y + xz \\ \dot{z} &= -\frac{8}{3}z + xy\end{aligned}\right\}.$$

This is a dynamical system and for any dynamical system we get a group of the character of the vector field. To simulate this problem or to do this we find the equilibrium and locally linearize around the equilibrium point and then we try to understand the behavior of around the equilibrium points. First we locate the equilibrium, obviously, (0, 0, 0) is one of them. But this is not enough for we have other parameters. Let us call the equilibrium points as.

$$\begin{aligned}A &= (0,0,0) \\ B &= \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right) \\ C &= \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)\end{aligned}$$

Note that if $r < 1$, points B and C will be imaginary and the position of the equilibrium is on the real plane. Therefore the two mentioned points do not exist for that range of r.

Now for the stability of the equilibrium points, which are obtained from the Jacobean matrix.

$$\sigma = 10,\ \beta = \frac{8}{3},\ r = 2$$

$$\begin{aligned}f_1(x, y, z) &= -10(x-y) \\ f_2(x, y, z) &= -xz - y + rx \\ f_3(x, y, z) &= xy - \frac{8}{3}z\end{aligned}$$

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & -x \\ y & x & -b \end{bmatrix}$$

Here we keep r as the variable parameter at (0,0,0) implies

$$\begin{bmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & \frac{-8}{3} \end{bmatrix} \tag{2}$$

Now we find the eigen value of (2) in terms of r, we obtain three eigen values,

$$\lambda = -b - 11 \pm \sqrt{\frac{81+40r}{2}} \tag{3}$$

Equation (3) is real for some values of r and for some it will not be real. Find for which value it will be stable. For r = 1, for instance the values will be, $\lambda$ = 0, $\lambda$ = −11

To further obtain the stability of the remaining to equilibrium point we use the Jacobian of system (1) above.

For r = 2

$$J = \begin{bmatrix} -10 & 10 & 0 \\ -1+2 & -1 & -\sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} & \sqrt{\frac{8}{3}} & \sqrt{\frac{8}{3}} \end{bmatrix}$$

The eigen values for this Jacobian is in terms of characters are one real negative(−ve) and two complex – conjugate, with negative( –ve) real part. The two equilibrium points are also called attractors just as a limit cycle is an attractor. The origin of the Lorenz equation will be the equilibrium point. We have seen earlier about the behavior when r = 1. After that, as the values of the parameter r is change through the value of r = 1, we have that particular point becomes unstable while two other equilibrium appears and they become stable. These are the equilibrium points B and C above. As r is changed further, we see that the real parts of the eigen values of those two equilibrium points having complex–conjugates (that is, they have inward spiraling orbits), while the real part will slowly go to zero.

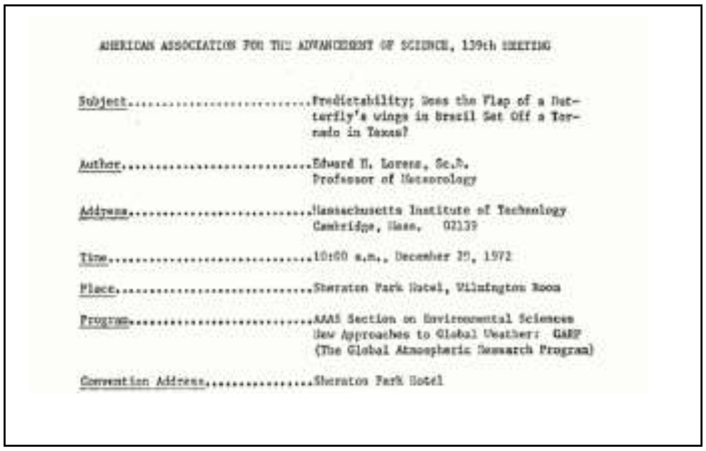
## DISCUSSION AND CONCLUSION

*"The trouble with weather forecasting is that it's right too often for us to ignore it and wrong too often for us to rely on it"*

-Patrick Young

Weather prediction is an extremely difficult problem. Meteorologists can predict the weather for short periods of time, a couple days at most, but beyond that predictions are generally poor.Why are weather predictions still so inaccurate? The root of the problem is something known as the "Butterfly Effect". Unfortunately, the Butterfly effect has been grossly misrepresented by modern culture. Another take on the butterfly effect is rooted in the following saying: "Does the flab of a Butterfly's wings in Brazil cause a hurricane in Texas?"

This Butterfly effect actually originated from a talk given in 1972 by Edward Lorenz, an MIT metreologist, see Fig. 1



AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, 139th MEETING

Subject..........................Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas?

Author..........................Edward N. Lorenz, Sc.D.
Professor of Meteorology

Address..........................Massachusetts Institute of Technology
Cambridge, Mass. 02139

Time..........................10:00 a.m., December 29, 1972

Place..........................Sheraton Park Hotel, Wilmington Room

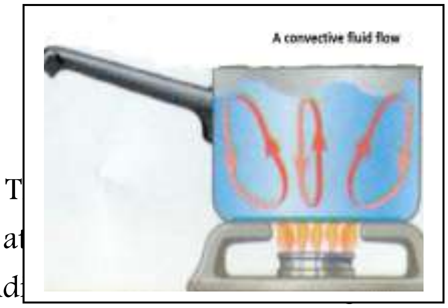
Program..........................AAAS Section on Environmental Sciences
New Approaches to Global Weather: GARP
(The Global Atmospheric Research Program)

Convention Address..........................Sheraton Park Hotel

**Fig. 1**

However, in the first sentence of the talk, Lorenz admitted the ridiculous nature of the title. The idea that the Butterfly effect could cause the hurricane is a bit absurd. That;

Thiscould cause This




Is a bit absurd

The root of Lorenz's idea was a central concept in Chaos T "Sensitivity to initial conditions". If we deviate a little and look a system called the "Double Pendulum". Sensitivity on initial cond pendulum in a slightly different position would cause drastically different behavior. The double pendulum is also an example of something called a "Dynamical System". Think of a dynamical system as simply a point in space, that point moves around as time passes by. For the double pendulum, this point would be the position of the pendulum tip. Sometimes, a dynamical system moves to and stays near a certain point is called an 'Attractor'. See Fig. 2 below

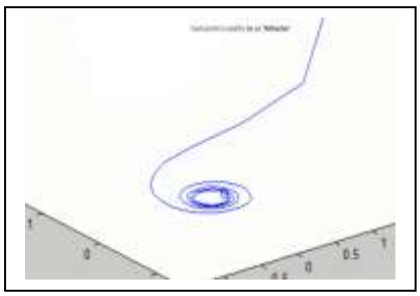


Fig. 2: An Attractor

Attractors can also be sets of points. For instance, a circle could be an attractor.

Now, going back to our main subject, the Lorenz Equation, as mentioned earlier, Edward Lorenz developed a set of equations to model a simplified weather system

$$\left.\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(\rho-z)-y\\ \frac{dz}{dt} &= xy-\beta z\end{aligned}\right\} \qquad (3.A)$$

Basically, the equations model the flow of fluid (particularly air) from a hot to a cold area. This is called a convective fluid flow. See Fig. 3 below

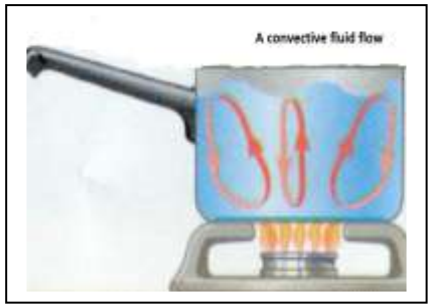



**Fig. 3: Convective Fluid Flow**

Convective flow in atmosphere can form super cool looking clouds, known as "roll clouds". This rolls from when air is heated from below and cooled from above. Remembering, of course, famous singings of geographers, that: the higher you go, the cooler it becomes. See Fig. 4 below.

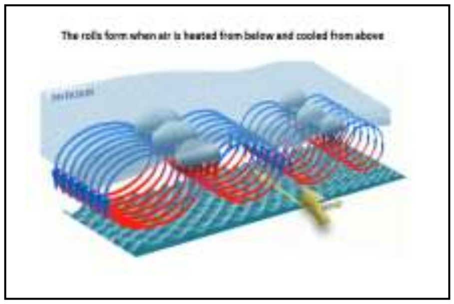



Fig. 4. Result of Convection

The final physical effect is displayed in the following figures.

**Fig. 5 Physical Results of Convection**

The solution to the set of differential equations to system (III.A) is a dynamical system. We employ the use of software called Matcont, a collection of toolboxes in MATLAB to plot the solutions with varying initial conditions. The result obtained through Matcont with the initial conditions $X(0) = (1, -1, 1)$, we get what we see in Fig. 6 below. The plot is sometimes called the "Lorenz Butterfly" due to its shape.

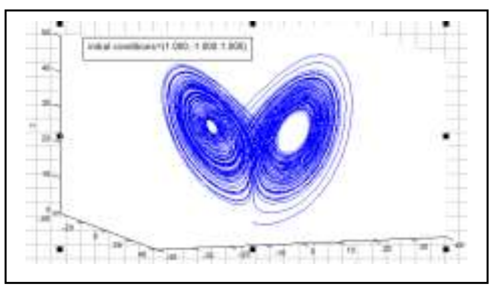


**Fig. 6 Lorenz Butterfly**

The plot seems to spiral around two distinct points, the attractors, one from the left and one from the right. The points are attractors, but of different sort than normal. They are also known as "strange attractors" Basically, this means that we cannot predict when a solution will jump from one attractor to another. We scaled the coordinate of the initial conditions by 10%, from 1 to 1.1, the solution may appear to be similar to the original, when the initial condition are $X(0) = (1, -1, 1)$, it is actually quite different, see Fig. 7 below. This becomes clear if we graph the distance between the two solutions, using the $l_2$-norm or the Euclidean distance

$$d_E(X,Y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$$

$$=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

It is a generalized form of the Pythagorean Theorem.

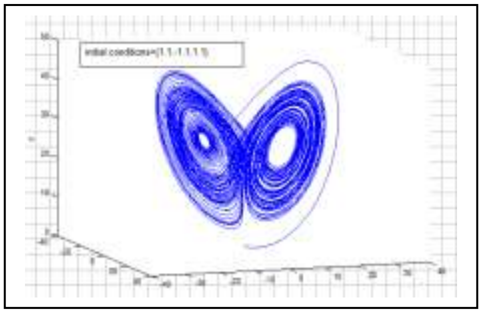


Fig. 7: Smaller Disturbance

If we think of 10% as being small, let us try a much smaller disturbance of the initial condition, 1%, so we have the Lorenz solution with initial conditions $X(0) = (1.01, -1.01, 1.01)$. Once again the graph looks similar to the previous two, but in fact quite different.

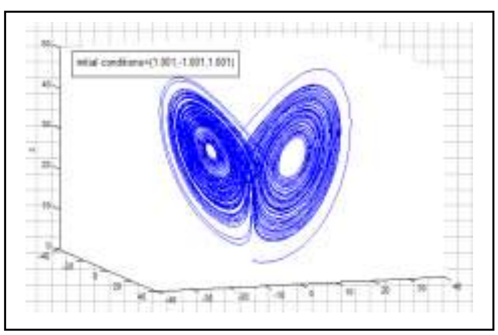


Fig. 8: Much Smaller Disturbances

If we further reduce the disturbance, all the four solutions plotted simultaneously, we see small differences between them as seen in Figure 9.

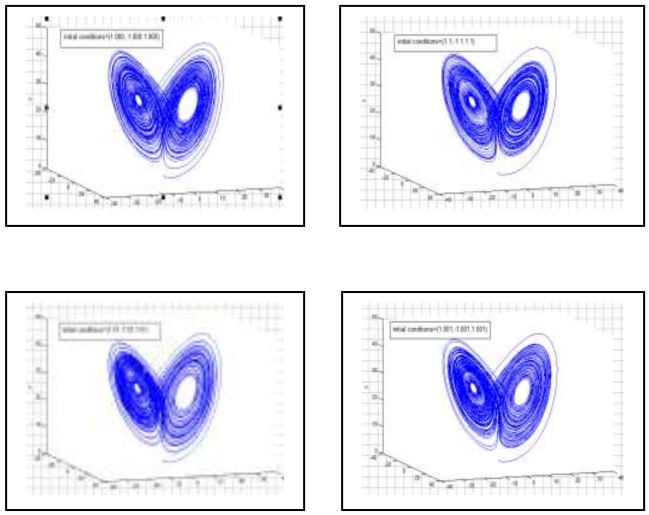


**Fig.9: All Disturbances Simultaneously**

## CONCLUSION

In conclusion, it is this sort of sensitivity on initial conditions that makes the weather difficult to predict. The simulation of Lorenz system is actually predictable accurately around 5 – 10 days. Predictions of weather longer than that cannot be accurate. If we manage to develop a better weather model, our predictions will be accurate for longer periods of time. No matter how good our model is, though, there will eventually be a point where it falls apart. Of course, the weather is far more than this simple set of Lorenz system given in (III.A).

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