Journal of Physical Science and Innovation **Volume 8, No. 1, 2016**
ISSN: 2277-0119

THE STRUCTURE OF THE PERMUTATION MODULES FOR TRANSITIVE ABELIAN GROUPS FOR PRIME-POWER-ORDER

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ABSTRACT

In this paper, we gave the well-known classification of transitive abelian groups of prime-power order before analyzing the structure of their permutation modules over fields of characteristic p. For a given prime number p, we have analysed the structure of the permutation module on characteristic p associated with transitive abelian p-groups of degree p^m ($m \geq 1$).

Keywords: *Permutation Modules, Transitive Abelian Group, Ascending and Descending Loewy Series.*

INTRODUCTION

Let G be a p-group acting transitively and faithfully on a set $\Omega$of size p^m, $m \geq 1$ and let F be a field of characteristic p. Define the descending Loewy series of the FG-module $F\Omega$as $F\Omega = J_0 > J_1 > J_2 > \ldots > J_{l-1} > J_l = \{0\}$ where for $0 \leq s \leq l$, J_{s+1} is the smallest submodule of J_s such that G act trivially on J_s/J_{s-1}. We also define the ascending Loewy series of $F\Omega$dually as $\{0\} = A_l < A_{l-1} < \ldots < A_1 < A_0 = F\Omega$where for $1 \leq s \leq l$, T_{s-1} is the largest submodule of $F\Omega$containing T_s such that G act trivially on T_{s-1}/T_s.

As an immediate consequence of the above definitions the following can de deduced:

1. The descending and ascending Loewy series of $F\Omega$ as we have already used in our notation, have the same length l. For the descending Loewy series $F\Omega = J_0 > J_1 > \ldots > J_{l-1} > J_l = \{0\}$ of $F\Omega$we have that for $1 \leq s \leq l$, J_s is the Jacobson radical of J_{s-1} and J_{l-1} is a one-dimensional submodule of $F\Omega$ (the so-called trace submodule); moreover J_1 is a submodule of co-dimension one (the augmentation).

2. Each term of the descending and ascending Loewy series of FΩis characteristic. Indeed fully invariant, in that if ψ X→ Y is a homomorphism of FG-modules, then $J_i(X)\Psi \leq J_i(Y)$; $A_i(X)\Psi \leq A_i(Y)$. In addition, for the descending Loewy series, if ψis subjective then $J_i(X)\Psi = J_i(Y)$.
3. Let J be the radical of FG where FG is considered as a module over itself. Then $J_s = (F\Omega)J^s$; moreover, for 1 ≤ s ≤ l, A_s is the submodule of FΩannihilated by J^s but not J^{s-1}.

Lemma1:

Let G be an abelian group of order p^n (n ≥ 1). Then G is a direct product of say k cyclic groups: $G = G_1 \times G_2 \times \ldots \times G_k$ where for some $g_i \in G_i$, we have that $G_i =< g_i >$ and g_i has order p^{n_i} , 1≤ i ≤ k; moreover $n = n_1 + n_2 + \ldots + n_k$.

Let F be a field of characteristic p. Then the group ring FG is isomorphic to the quotient ring $R = F[x_1, x_2, \ldots, x_k]/(x_1^{p^{n_1}}, x_2^{p^{n_2}}, \ldots, x_k^{p^{n_k}})$ of polynomial ring $F[x_1, x_2, \ldots, x_k]$ over F, where $(x_1^{p^{n_1}}, x_2^{p^{n_2}}, \ldots x_k^{p^{n_k}})$ is the ideal generated by the set $\{x_i^{p^{n_i}} : 1 \leq i \leq k\}$.

Proof: The first part is a fundamental theorem of abelian groups. As such, we only need to prove the second part.

Let $\{x_1, x_2, \ldots, x_k\}$ be distinct indeterminate over F. The mapping $\Psi : x_i \to g_i - 1$, 1 ≤ i ≤ k can be extended to a ring homomorphism ψ R→FG, as can be verified quite readily. Also ψ is subjective since for each i with 1 ≤ i ≤ k we have that $\Psi(x_i + 1) = \Psi(x_i) + \Psi(1) = g_i - 1 + 1 = g_i$. And as F has characteristics p, we see that $\Psi(x_i^{p^{n_i}}) = \Psi(g_i - 1)^{p^{n_i}} = g_i^{p^{n_i}} - 1 = 1 - 1 = 0$. Thus the ideal $(x_1^{p^{n_1}}, x_2^{p^{n_2}}, \ldots, x_k^{p^{n_k}})$ is in the kernel of ψ Furthermore, the monomials $\{x_1^{i_1}, x_2^{i_2}, \ldots, x_k^{i_k} : 0 \leq i_j \leq p^{n_j}\}$ are linearly independent and they span R. Thus they form a basis for R over F. Moreover since there are p^{n_j} choices for each of the i_j, we have that $\dim R = \prod_{1 \leq i_j \leq k} p^{n_j} = p^n$. But also dimFG = p^n. That is R and FG has the same dimension and accordingly R ≅ FG.

The Classification of Transitive Abelian P-Groups

We find it necessary to give the following well-known classification of transitive abelian groups of prime-power order before analyzing the structure of their permutation modules over fields of characteristic p. Let Ωdenote a set of size p^m ($m \geq 1$) and G be a p-subgroup of Sym (Ω); G is always faithful on Ω Let G act transitively on Ωdenote the centre of G by Z. Observe that G is semi-regular on $\Omega$and so its order is equal to the size of its orbits. Accordingly $|Z|$ is at most p^m. When G is abelian it is evident that G act regular on $\Omega$and the centre of G has order p^m. Conversely, suppose that $|Z| = p^m$. Choose and fix an arbitrary element $\alpha \in \Omega$and define $L = G_\alpha$. By the faithfulness of G on Ωno non-identity normal subgroup of G is contained in L. Thus $L \cap Z = 1$ and we may form $L \times Z$. By transitivity of G, we have that $p^m = |G:L| = |Z|$ and so $G = L \times Z$. This mean that L is normal in G and as such $L = 1$. Consequently $G = Z$. We have, therefore, shown that G is abelian if and only if its centre has order p^m. Now the number of different abelian groups of order p^m up to isomorphism, is the number of partition $\pi(m)$ of m. But G must be one of these acting regularly. Therefore, there are, up to equivalence, $\pi(m)$ distinct different number of faithful transitive p-groups of degree p^m whose center has order p^m. And certainly

$\pi(m) < m$ if $m \leq 3$ and $\pi(m) > m$ if $m \geq 4$.

The Permutation Modules for Transitive Abelian P-Groups

For the remainder of this paper, G is an abelian p-group, $\Omega$is transitive G-space of size p^m ($m \geq 1$), and F is a field of character p. We prove that the ascending Loewy series of $F\Omega$coincides with its descending Loewy series. The detailed descriptions of these series are also given. We begin by setting up some useful notation. Since G is abelian and transitive it follows that G is regular on Ω So $|G| = |\Omega| = p^m$ and we may identify G with Ω(and hence FG with $F\Omega$). Thus we shall henceforth talk of the ascending and descending Loewy series of FG in place of those of $F\Omega$ Also, using lemma 1, we see that G is the direct product of say k cyclic groups, $G = G_1 \times G_2 \times \ldots \times G_k$ where for each i with $1 \leq i \leq k$ we have that $G_i = < g_i >$ for some g_i in G_i and g_i has order p^{m_i}; moreover $m = m_1 + m_2 + \ldots + m_k$. Also following the same lemma , we may think of FG as the ring of polynomials: $\sum_{i_1}\sum_{i_2}\ldots\sum_{i_k} a_{i_1,i_2,\ldots,i_k} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$, where $a_{i_1,i_2,\ldots,i_k}$ are in F and the variables x_i are some distinct indeterminate over F satisfying the conditions $x_i^{p^{m_i}} = 0$, $1 \leq i \leq k$. Furthermore, the action of g_i on $F\Omega$is by multiplication $(1 + x_i)$.

In view of the set up, we prove the following result

Theorem

Let G be an abelian p-group acting transitively on a set Ωof size p^m (m ≥ 1) and let F be a field of characteristic p. Then the ascending and descending Loewy series of FΩcoincide. Moreover, these series coincide $F\Omega = A_0 > A_1 > A_2 > \ldots > A_{l-1} > A_l = \{0\}$ where $A_r = span(x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} : 0 \le i_j < p^{m_j}, \sum i_j \ge r)$ for all r such that 0 ≤ r ≤ l–1; and $l = 1 + \sum_{i=1}^{k} (p^{m_i} - 1)$. As such the Loewy length of FΩis l. For the proof we require a lemma.

Lemma 2:

The descending Loewy series of FΩis $F\Omega = A_0 > A_1 > A_2 > \ldots > A_{l-1} > A_l = \{0\}$.

Proof of Lemma 2:

Indeed let r be arbitrary but fixed (0 ≤ r ≤ l–1) and denote J_{r+1} the radical of A_r. Choose any element of f in A_r. For any $g_i (1 \le i \le k)$ we have that $fg_i = f(1 + x_i) = f + fx_i$. But fx_i lies in A_{r+1}. Thus $(f + A_{r+1})g_i = f + A_{r+1}$; and so since the g_i generate G, G acts trivially on A_r / A_{r+1}. By definition J_{r+1} is the smallest submodule of A_r such that G acts trivially on A_r / J_{r+1}. As a result we see that $J_{r+1} \le A_{r+1}$. Now for each g_i and for any f in A_r we have that $(f + J_{r+1})g_i = f + J_{r+1}$. Therefore, each ($g_i$–1) acts as a zero on f. That is f (g_i–1) lies in J_{r+1}. In other words fx_i is in J_{r+1} and this holds for all x_i, 1≤i≤ k. Thus $A_{r+1} = J_{r+1}$ (0 ≤ r ≤ l–1). Furthermore, by definition; $A_{l-1} = span\{x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} : i_1 + i_2 + \ldots + i_k \ge l - 1\}$, and as each i_j is less than or equal to $(p^{m_j} - 1)$ we must have that $i_j = p^{m_j} - 1$, 1≤i≤k. As such, A_{l-1} $span\{x_1^{p^{m_1}-1} \ldots x_k^{p^{m_k}-1}\}$ and A_l = {0}. This proves the lemma.

Proof of Theorem

Recall that the descending and the ascending Loewy length of F are equal. So, let $\{0\} = M_l < M_{l-1} < \ldots < M_2 < M_1 < M_0 = F\Omega$ be the ascending Loewy series of FΩ To prove our theorem we need to show that M_r = A_r for all r such that 1 ≤ r ≤ l–1. Let $f = \sum_{i_1} \sum_{i_2} \ldots \sum_{i_k} a_{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$ be in M_{l-1}. Then, by definition $fg_i = f$ for any g_i, 1≤ i≤ k. Therefore $f(g_i - 1) = 0$. That is $fx_i = 0$ for all i, 1 ≤ i ≤ k; and consequently f belongs to $span\{x_1^{p^{m_1}-1} x_2^{p^{m_2}-1} \ldots x_k^{p^{m_k}-1}\}$ since the gi generate G. Thus $M_{l-1} \le A_{l-1}$. By definition, M_{l-1} is the largest non-zero submodules of FΩon which G acts trivially. Hence we also have that $A_{l-1} \le M_{l-1}$. As such M_{l-1} = A_{l-1}. Now suppose that A_r = M_r for some 2 ≤ r ≤ l–1. Then by definition, Ar_{-1} ≤ M_{r-1}. Let f = $\sum_{i_1} \sum_{i_2} \ldots \sum_{i_k} a_{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$ be in M_{r-1}. Then for each g_j with 1 ≤ j ≤ k, we have that $f(g_j - 1) \in M_r$; that is $fx_j \in M_j$ for all j, 1≤ j ≤ k. Thus for an arbitrary but fixed j with 1 ≤ j ≤ k, we have that $\sum_{s=1}^{k}(i_s + \delta_{js}) \geq r$. That is $(1 + \sum_{s=1}^{k} i_s) \geq r$. Accordingly, $(\sum_{s=1}^{k} i_s) \geq r - 1$. This show that f is in A_{r-1} and so $M_{r-1} \leq A_{r-1}$. Hence $M_{r-1} = A_{r-1}$. The theorem now holds by induction.

We now obtain some relationship between the dimensions of the factors of the Loewy series of the FG-module F. All our notation will be as used for any r with 0 ≤ r ≤ l, define $H_r = span\{x_1^{i_1} x_2^{i_2} ... x_k^{i_k} : 0 \leq i_j \leq p^{m_j} and \sum_{j=1}^{k} i_j = r\}$. Then H_r is the space of homogeneous polynomial of degree r and $A_r = \oplus_{j \geq r} H_j$. Moreover, $H_{r-1} = A_{r-1} / A_r$ (1 ≤ r ≤ l). The FG-module, FΩ is self-dual and dualising its descending Loewy series gives us its ascending Loewy series. These two series coincides. Hence, we must have that the dual, A_r^* of A_r coincide with A_{l-r} (1 ≤ r ≤ l-1). Thus $A_r \cong A_r^* \cong A_{l-r}$ and as such $A_r / A_{r+1} \cong A_{l-r-1} / A_{l-r}$. That is $H_r \cong H_{l-r-1}$ (0 ≤ r ≤ l). Accordingly, we obtain the following corollary to theorem.

Corollary

Let $\{H_r : 0 \leq r \leq l\}$ be sequence of factors of the Loewy series of FΩand define $d_i = \dim H_{i-1}$, 1≤ i ≤ l. Then $d_i = d_{l-i+1}$.

CONCLUSION

Suppose that in the Theorem we have the special case: $M_i = 1$ for all i such that 1≤ i ≤ k. Then k = m and G becomes an elementary abelian p-group of rank m. In addition the Loewy series of the group-ring FG is $FG = J_0 > J_1 > J_2 > ... > J_l = \{0\}$

where $J_r = span\{x_1^{i_1} x_2^{i_2} ... x_k^{i_k} : 0 \leq i_j \leq p-1, 1 \leq j \leq m and \sum i_j \geq r\}$ for all r such that 0 ≤ r ≤ l-1 and

l = 1 + m(p − 1). Conversely, suppose G is a P-group of order pm and F is a field of characteristic p. If the Loewy length of the group-ring FG is 1+m(p-1) then G is elementary abelian. The problem of describing the descending and the ascending Loewy series of the permutation modules of non-abelian transitive p-groups in characteristic p is not easily solve. Perhaps the best way to go round it is to consider groups of small degree.

REFERENCE

Audu, M.S. (1988b). The Structure of Permutation of Modules for Transitive p-Groups of Degree p^2. Journal of Algebra Vol. 117:227-239.

Audu, M.S. (1991). The Loewy Series Associated with Transitive P-Groups of Degree p^{2n}. ABACUS, Volume 21, No.2, pp 1 – 9.

Audu, M.S. and Momoh, S.U. (1996). The Number of Transitive P-Group of Degree p^{3n}. *Spectrum Journal,* Vol. 3 No. 1 and 2, pp 155- 159.

Bamberg, J. and Praeger, C.E. (2004). Finite Permutation Groups with Transitive Minimal Normal Subgroups. Product London Mathematical Society, 3 Vol. 89(1): 71-103.

Cameron, P.J., Kovacs, L.G., Newman, M.F. and Praeger, C.E. (1985). Fixed-Point-Free Permutation in Transitive Permutation Groups of Prime-Power-Order. *Quarterly Journal of Mathematics Oxford.* Vol. 2(36). 273-278.

Jennings, S.A. (1941). The Structure of the Group-Rings of a P-Group over a Modular Field. Trans. Amer. Math. Soc. Vol. 50, pp 175 – 185.

Tsushima, Y. (1979). Some Notes on the Radical of a Finite Group-ring II. *OSAKA Journal of Mathematics.* Vol. 16. Pp 35 – 38.

Wielandt, H. (1964). Finite Permutation Groups. Academic Press, New York.

Reference to this paper should be made as follows: Mohammed A., et al. (2016), The Structure of the Permutation Modules for Transitive Abelian Groups for Prime-Power-Order. *J. of Physical Science and Innovation, Vol. 8, No. 1, Pp. 16 – 20.*

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