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A FINITE SUM OF HETEROGENEOUS SELF-SIMILAR CANTOR-LIKE SET

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ABSTRACT

A finite sum of heterogeneous cantor like set was considered. it was proved that this set is admissible by certain finite family of uniformly contracting self-similar set which is more general than the one considered by P. Mendes and a succinct prove is given under a weaker condition than the one in theorem 2 of P. Mendes.MSC (2010): Primary 28A80, 28A78; Secondary 54C30.

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INTRODUCTION

Let (X, ρ) be a metric space and $\rho(x, y)$ the distance between x and y, i.e. $\rho(x, y) = \sup\{|x - y|: x, y \in X\}$. A map $f: (X, \rho) \to (X, \rho)$ is called a contraction if their exist a number $c \in (0,1)$ such that

$$\rho(f(x), j\,(y)) \leq c\rho\,(x, y) \text{ for all } x, y = \in (X, \rho) \qquad (1.1)$$

For simplicity we write X for (X, ρ)
Let$E \subset X$, then E is said to be self-similar if fj $(j = 1,2, \dots, k)$is such that

$$E = \bigcup_{j=1}^{k} fi(E) \qquad (1.2)$$

The set E constitute an attractor for an iterative function system (denoted IFS) such that $f1, \dots, fk$ constituting the IFS are similarities with the contraction ratio $c1 \dots, ck$ provided $cj < 1$ $for\ all\ j = 1, \dots, k$. By the IFS theory [1,2,4,6], we are guaranteed the existence of a nonempty compact set in X (since X is complete) that satisfies (1.2).In general, an affine map is a linear transformation, so that it is easy to see that a self-similarity is a particular case of affine map Relating to self-similarity is the concept of "fractal" which appeared for the first time in B. Mendelbrot's book [5]. It contribution consists in revealing common features behind objects and shapes that can be found in nature. One of these feature was that of self-similarity. The Cantor ternary set is one such example of a self-similarity set[1,2,4,6]. It is on the set of this nature Hausdorff introduced the study of fractal dimensions in the twenties after the work of B. Mandelbrot. Pedro Mendes related the conjecture "if the sum of two affine cantor sets has positive lebesques measure, then "contains an interval" with the question "if it is true that if a self-similar set has positive lebesque measure, then it contains an interval?" by proving the following theory.

Theorem 1.1: There is dense set
$D \in H(2r+1) \times H(2r+1)$, such that if $(\bar{\lambda}, \bar{y}) \in D$, then $C(\bar{\lambda}) + C(\bar{y}) = \{x + y: x \in C(\bar{\lambda}), y \in C(\bar{y})\}$ is a uniformly contracting self-similar set.

In the theory above, P. Mendes [6] considered the sum of two homogeneous cantor set with a common contraction ratio. The purpose of this paper is to consider the sum of finite families of "heterogeneous" canter sets which is of more general form than the one in [6]. For more on this conjecture see [7,9,10].

Preliminaries

For every arbitrary but fixed $m \in N$, let $f: R \to R$ be given by

$$f_{mj}(x) = \lambda_x x + a_{m,j;}\ j = 1,2,\dots,k$$

be an affine contraction on R.

Where $\lambda_m \in (0,1)$, $\quad a_{m,j} = \lambda_m + \lambda_{m,1} + \lambda_m + \lambda_{m,2} + \cdots + \lambda_{m,k} + \lambda_m = 1.$ *Let*

$$H^{2mk+m} = \{\bar{U} \in R^{2mk+m}: \bar{U} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)\}$$

$$Where\ \bar{u}_i = (\lambda_i, \lambda_{i1,}\lambda_i, \lambda_{i,2,} \dots \lambda_{i,k,}\lambda_i) \in R^{2k+1} for\ i = 1,2,\dots,m.$$

Let $C(\bar{\lambda}_i)$ be self similar set obtained for each I using the contractions above. We call the set $C(\bar{\lambda}_i)$ for i = 1,2,...,m, m-summable heterogeneous cantor set, uniformly homogeneous m-summable cantor set if $\bar{\lambda}_i = \bar{\lambda} \forall_i$ and simply uniformly homogeneous 2-summable cantor setif $\bar{\lambda} i = \bar{\lambda}\ \forall I, m = 2$.

Let $S: A \subset [a,b] \to B \subset [c,d]$ $(a, b, c, d \in R;\ a < b, c < d)$be an affine bijection, it is easy to see that their exist $g: A \subset [a,b] \rightarrow R$ and $g: B \subset [c,d] \rightarrow R$ such that $f = S-1 \text{ o } g \text{ o } S$ and $g = S \text{ o } f \text{ o } S-1$ [3].

Let $Am \{am,j : j = 1,2,\ldots,k\}$ and $Bm = \{bm,j : j = 1,2,\ldots,k\}$ such that $am,j < am,j+1, am,j > am+1,j$ and $bm,j < bm,j+1, > bm+1,j$ for all m,j

Now, let $g_{i,j}: R \rightarrow R$ be such that $g_{i,j}(x) = \mu_{ix} + b^{i,j}$ ($j = 1,2,\ldots,k$; $i = m,m+1,\ldots,m+r-1$).

Then clearly

$$S(x) = \frac{b_{m,r} - b_{m,1}}{a_{m,r} - a_{m,1}}x + \frac{a_{m,r}b_{m,1} - a_{m,1}b_{m,r}}{a_{m,r} - a_{m,1}} \quad (2.1)$$

Is an affine bijection on R. So that by the existance of $f = S^{-1}$ o g oS, we have that

$$\left.\begin{array}{c} f_{i,j}(x) = \mu_i x + a_{i,j}, \mu_i \in (0,1) \forall i \\ a_{i,j} = \dfrac{a_{m,1}(b_{m,1} - b_{i,j}) a_{m,r}(b_{i,j} - b_{m,1})}{b_{m,r} - b_{m,1}} \end{array}\right\} \quad (2.2)$$

Let $\mu = \sup < 1$, then $f_{i,j}, g_{i,j}$ is an arrays of uniformly contracting affine

$$m \le i \le m + r - 1$$

Maps, consequently there exist a unique nonempty conpact subsets of R, A, B such that $A = Ui,jfi,j, B\ Ui,jgi,j$ for every i called the uniformly contracting self-similar Set.

Now, constructively, let $A0 = [0,1]$ and I be a countable set with at least two elements. Put $IN = \{j1, j2, \ldots)\ j\rho : j\rho \in I, \rho \in N\}$ and $IN|Nn = \{(j1, j2, \ldots jn) : j\rho \in I, \rho \in Nn\}$. Then for every fixed $m \in N$, observe that $fm, ji, j2, \ldots jn\ (A0)$ is an interval of length μ_m^n and if we put

$$A_{m,n} = \bigcup_{p=1}^{n} f_{m,ji,j2,\ldots jp}\,(A_0);\ \forall\ fixed\ m$$

Where Am,n is the nth step construction of the set A for a fixed $m \in N$, As i varies over m we have that

$$A_{i,n} = \bigcup_{p=1}^{n} f_{m,ji,j2,\ldots jp(A_0)\ i=1,2,..,m}$$

Observe that $Ai,n+1 \subset Ai,n\ \forall\ n > 1$, hence there is a unique $A_{i,*}$ for the sequence of set $\{Ai,n\}n > 1$ such that

$$A_{i,*} = \bigcap_{n=1}^{\infty}\bigcup_{p=1}^{n} f_{m,ji,j2,\ldots jp}\,(A_0);\ i=1,2,..,m.$$

Now, if we put $A_{m,*} =$

$\cup_{i=1}^{m} A_{i,*}$, similarly observe that $A_{i,*} \subset A_{m+1,*}, \forall\ m \le 1$ consequently, there is a unique $A_{*,*}\ A \subset A_0$ for the sequence of set $(A_{i,n})_{n \ge 1}$ such that

$$A_{*,*} = \bigcap_{m=1}^{\infty}\bigcup_{i=1}^{m} A_{i,*,}.$$

All together we have that

$$A = \bigcap_{m=1}^{\infty}\bigcup_{i=1}^{m}\left(\bigcap_{n=1}^{\infty}\bigcup_{p=1}^{n} f_{m,j1,j2,\ldots jp}(A_0)\right)$$

In particular, if m = 1, we obtain $A_{i,*}$, the case of the work in [3], and if n = 1, we obtain $A_{*,1}$ which is a different consideration. Furthermore observe that $A_{1,*}$and $A_{*,1}$ are special cases of our construct $A_{*,*}$ (i.e. A).

$$\text{Let } \alpha \in A;\Rightarrow \alpha \in A_m \forall m \leq 1; \Rightarrow \alpha \in \bigcup_{i=1}^{m} A_{i,*;} \Rightarrow \exists n_0 \in \mathbb{N}, j_{n0} \in I$$

such that $\alpha \in A_{i,j1,j2,\ldots jn0} \forall i = 1,2,\ldots,m.; p \geq n_0;$

$$\Rightarrow \alpha \in \cap_{n0\geq 1}\cup_{p\geq n0} f_{i,ji,j2,\ldots jp}(A_0) \forall i = 1,2,\ldots,m.$$

$$\Rightarrow \exists;_{ai,jp} \in \{a_{m,j}:: j = 1,2,\ldots,k.; i = 1,2,\ldots,m\}\ such\ that$$

$\alpha = \Sigma_{p\geq 1\ ai,jp}\mu_i^p \Sigma_{p\geq 1\ ai,jp}\mu_i^p \forall\ i$

At this juncture we consider the strong open set condition (SOSC) which was introduced by P. Moran [8]. Let the collection $f_{i,j}$ be as defined above, then the set A is said to satisfy the SOSC if there exists a nonempty bounded open set U ⊃ A such that for every fixed i, $f_{i,j}$ (U) ⊂ U g for j = 1,2,…, k and $f_{i,j}$ (U) ∩$f_{i,\ell}$ (U) = ø for j = ≠ ℓ. If an addition, $f_{i,j}$ satisfies $f_{i,j}$ (U) ⊂ U for i = 1,2,…, m; j = 1,2,…, k; $f_{i,j}$ (U) ∩$f_{s,j}$ (U) ≠ ø for some j, i, s = 1,2,…, m and $f_{i,j}$ (U) ∩$f_{i,j}$ (u) = ø for some j, i, s = 1,2,…, m. We shall call this a weak open set condition (WOSC). Observe that the WOSC implies the SOSC, however, the reverse may not be necessarily true. It is well known that if a self-similar set A satisfies SOSC, then A has Hausedorff dimension less than one and positive Hausdorff measure. We adopt the terminology in [3] and call the interval $L_{i,j}$ = ($f_{i,j}$(1), $f_{i,j}$(0)) (i = 1,2,…, m; j = 1,2,…, k – 1) a lake, then we define F = {J ⊂ [0,1] centered in A; J ⊄ $f_{i,j}$ ([0,1]), j = 1,2,…,k, for fixed i}.

Main result

Theorem 3.1: There is dense set D

$\in \quad \Pi_{i=1}^{m} H_i^{(2k+1)}, such\ that\ if\ \bar{U} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \in D, then\ \sum_{i=1}^{m} C(\lambda_i) = \{\sum_{i=1}^{m} x_i \in C(\lambda_i);\ i = 1,2,\ldots,m\}$

is a uniformly contracting m-summable heterogeneous self-similar set.

Proof

$$\bar{U} \in D \iff (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \in H^{2k+1}; i = 1,2,\ldots,m$$

But $\bar{u}_i = (\lambda_i, \lambda_{i,1}, \lambda_i, \lambda_{i,2}, \ldots, \lambda_{i,k}, \lambda_i) \in H^{2k+1}$ for i = 1,2,…, m.

This implies there exists $\beta \in (0,1)$ and $\rho_I \in N$ such that $\lambda_i = \beta^{pi}$; i = 1,2,…,m.
Now for a fixed i_0, $\lambda_{i0} \in (0,1)$, define $f(t) = \lambda_{i0}^{t}$ then $f : (0, \infty) \rightarrow (0,1)$ is a continuous decreasing function. Thus if $\lambda_{i0,j}$ is a small perturbation in $\bar{u}_{10}$, then we approximate the element $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \in \Pi_{i=1}^{m} H_i^{(2k+1)}$ by an element of the form

$$\bar{u}_1 = (\lambda_{i0}^{P_i/pio}, \lambda_{i,1}, \lambda_{i0}^{P_i/pio}, \lambda_{i,2,\ldots,} \lambda_{i,k}, \lambda_{i0}^{P_i/pio}); i = 1,2 \ldots, m$$

Thus, the choice of $\beta = \lambda_{i0}^{P_i/pio}$ we have that $\bar{U} \in D$.
Now, if $x_i \in C(\lambda_i)$ then there exists $a_{r,jn}$ such that

$$x_i = \sum_{n \geq 1} a_{r,jn} \lambda_i^n = \sum_{n \geq 1} a_{r,jn} \beta^{npi}$$

Then $\sum_{i=1}^{m} x_i$ is given by a power series in $\mu = \beta \prod_{i=1}^{m} p_i$, whose coefficient vary in the finite set of all sums of the form $\sum_{i=1}^{m} \sum_{n=1}^{pi-1} a_{r,jn} \beta^{npi}$. By the above construction, it follows that $\sum_{i=1}^{m} C(\lambda_i)$ is a uniformly contracting m-summable heterogeneous self-similar set.

Remark

It suffice to choose β so that $\beta = \sup\{\lambda i\ i = 1,2, \ldots, m\}$. Also the k used in our construction depend on i, however for simplicity we write k instead of k (i).

Theorem If F ≠ ø then the uniform contracting m-summable heterogeneous self-similar set A has Hausdorff dimension less than one and positive hausdorff measure at this dimension for all but fixed i ∈ {1,2,…,m}.

Proof

Assume $F = \emptyset$ ($\nexists$ $J \subset [0,1,]$ such that $J \not\subset f_{i,j}([0,1]), j = 1,2, \ldots, k$
Observe that $f_{i,1}(0) 0$ and $= f_{i,k}(1) = 1\ \forall\ i = 1,2, \ldots, m$. Now for a fixed $i_0 \in$ {1,2,…,m}, since $f_{i,j}$ is a contraction map, it implies there exists, $J_{i,j}$ such that $f_{i0,j}(A) = J_{i0,j} \subset A$; j = 1,2,…,k, where $J_{i0,j} = (f_{i0,j}(0), f_{i0,j}(1))$. Since A is an attractor for the IFS, we shall have

$$A \supset \bigcup_{j=1}^{k} J_{io,j} = \bigcup_{j=1}^{k} f_{io,j}(A) = A$$

Thus it is easy to see that $L_{i0,I} \subset [f_{i0,j}(0), f_{i0,j}(1)]$ which implies that $L_{i0,j} \subset [0,1]$ and $L_{i0,j} \not\subset f_{i0,j}([0,1]) \Rightarrow\Leftarrow$ which is a contradiction to the assumption that $F = \emptyset$

Since $L_{i0,j}$ separates $J_{i0,j}$ and $J_{i0,j+1}$ j = 1,2,..., k so that $J_{i0,j} \cap J_{i0,j+1} = \emptyset$ $\forall$ j = 1,2,..., k which implies that $f_{i0,j} \cap f_{i0,j} = \emptyset$

Inductively, going by our construction we have that
$f_{i0,j} \cap f_{i0,j} = \emptyset$; $j \neq t$, $\forall$ $i_0 \in \{1,2,...,m\}$Thus, the self-similar heterogeneous set A satisfies the SOSC, hence has Hausdorff dimension less than one and positive Hausdorff measure at this dimension.

Remark

The method of our proof for theorem 3.3, I believe is far much simple than the one by Pedro [6]. Furthermore, the assumption f (c) ≠ 0 was dispensed with. Some of the restriction imposed on a_i in [6] was droped, I believe this should be taken care of naturally by ones method of construction.

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